TEST PAPERS IN GEOMETRY

FOR THE USE OF CANDIDATES PREPARING FOR SCHOOL CERTIFICATE, MATRICULATION AND SIMILAR EXAMINATIONS

BY

W. E. PATERSON, M.A., B.Sc.

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PREFACE

This collection of papers is intended to provide tests of the same standard as school certificate examinations, with questions of the same varied nature. They are not graded, and will probably be found to vary in difficulty just as actual examination papers vary from year to year.

Book work is included in the papers, partly to provide the necessary revision, but also to give some clue to the solution of the following rider.

It is advised that care and attention be given to questions requiring figures drawn to scale; facility in the accurate use of mathematical instruments is particularly important for the examination candidate who has no great skill in solving riders. When the figure is not required to be drawn to scale, it wastes time to use instruments. Figures for propositions and riders should be drawn freehand, but should be neat, of a fair size, and reasonably accurate.

The first fifty papers are limited to the syllabus of the Matriculation and School Certificate Examinations of the University of London; to meet the requirements of other examinations the last fifty papers include questions involving proportion and similarity, and questions needing elementary trigonometry. Further practice in trigonometrical manipulation can be obtained by solving scale-drawing questions by calculation.

A special edition is published, which gives the answers to numerical questions and skeleton solutions to all but the simplest riders. Proofs are given of many pieces of book work which are included in some textbooks of geometry but not in all.

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TEST PAPERS IN GEOMETRY

No. 1

1. In a triangle ABC the angle ABC is $120°$; equilateral triangles DAB, EBC are described on the sides AB and BC, so that they fall entirely outside the triangle ABC. Prove that $DE = AC$.

2. Define a parallelogram and prove that the diagonals bisect one another.

State and prove additional properties of the diagonals of (i) a rectangle, (ii) a rhombus.

3. Construct a triangle ABC, having $AB = 4{\cdot}7$ cm., $AC = 3{\cdot}4$ cm., and angle $BAC = 75°$. Draw the perpendicular bisector of BC and the bisector of the angle BAC. Call their point of intersection D, and measure the angle BDC.

4. State and prove the geometrical proposition corresponding to the algebraical formula for $(a + b)^2$.

In a triangle ABC with a right angle at C, a perpendicular CD is let fall on AB. Prove that $CD^2 = AD \, . \, DB$.

5. Two straight lines AX, AY contain an angle of $55°$, and a circle of radius 3 cm. rolls along XA towards A.

Find by an accurate drawing the distance of the centre of the circle from A when the circle touches AY. Give your reasons.

6. A circle is drawn having the side BC of an acute-angled triangle ABC as diameter. Prove that A must be outside the circle.

If the circle cuts AB at P, and AC at Q, and if BQ, CP cut at X, prove that AX is perpendicular to BC.

No 2

1 If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles opposite to one pair of equal sides equal, show that the angles opposite the other pair of equal sides are either equal or supplementary

If the bisector of the angle A of a triangle ABC also bisects the side BC, prove that $AB = AC$

2 If two non parallel lines are cut by a transversal, prove, without using any properties of parallel lines, that the two interior angles on that side of the transversal on which the given lines tend to meet are together less than two right angles

State the converse of this

✓3 Draw a triangle ABC, being given that $AB = 6\ 2$ cm, $AC = 5\ 7$ cm, and AD, the perpendicular from A to BC, $= 3\ 8$ cm Give reasons for your construction and show that there are two solutions

4 If a straight line be divided into two parts, show that the sum of the squares on the whole line and one part is equal to twice the rectangle contained by the whole line and that part together with the square on the other part

In a square $ABCD$ A is joined to any point E in BC, prove that $AE^2 = 2\ BC\ \ BE + CE^2$

5 Prove that the perpendicular from the centre of any circle on any chord of the circle bisects the chord

Take a point A 1 7 in from the centre of a circle of radius 2 in Through A draw a chord BC, such that BC is bisected at A, also draw through A a chord PQ of length 3 in Prove your constructions

6 A statue BC of height 8 ft stands on a pedestal AB of height 11 ft, a man whose eye is 5 ft from the ground walks along a path which is at right angles to AB Find two positions of his eye E such that the angle $BEC = 20°$, and find, by measurement, the distance between the positions (Take 1 cm to represent 1 ft)

No. 3

1. Two triangles have their angles equal, each to each, but they have no sides equal; prove that the small triangle may be placed so that two of its sides fall on two sides of the large triangle, and that the third sides are then parallel.

2. What are the respective loci of a point when it moves under the following conditions—

(i) So that it is always 6 in. from a fixed point;
(ii) So that it is always 6 in. from a fixed straight line;
(iii) So that the lines joining it to two fixed points always contain an angle 90°?

3. Draw a parallelogram with diagonals 2·3 in. and 1·5 in. long, and one side 1·1 in. long. Measure the obtuse angle contained by the diagonals.

4. Prove Apollonius's theorem, viz.—

The sum of the squares on two sides of a triangle is equal to twice the square on half the third side together, with twice the square on the median bisecting that side.

Two points A and B are 2 in. apart, draw the locus of a point P which moves so that $PA^2 + PB^2 = 6\frac{1}{2}$ sq. in.

5. Prove that the radius of the inscribed circle of a triangle is equal to the area of the triangle divided by its semi-perimeter.

What is the diameter of the largest circular pond that can be made in a triangular field whose sides are 65 yds., 56 yds., 33 yds.?

6. From a point P a tangent PA is drawn to a circle and a secant PBC; from the secant PD is cut equal to PA. Prove that AD bisects the angle BAC.

No. 4

1 The bisector of the opposite angles A and C of a parallelogram $ABCD$ meet the diagonal BD at E and F respectively Prove that $AE = CF$

2 Two roads AB, AC cross at an angle of 75°, B a mile stone is 800 yds from A It is required to place a flagstaff equidistant from A and B and equidistant from the two roads By drawing a figure to scale find the distance of the flagstaff from A

3 Prove that the straight line joining the middle points of two sides of a triangle is parallel to the third side

If the middle points of the consecutive sides of any quadrilateral are joined, show that the quadrilateral so formed is always a parallelogram and that it is equal to half the original quadrilateral

4 From the right angle A of a triangle ABC a perpendicular AD is let fall on the hypotenuse BC, prove that the square on AB is equal to the rectangle BC, BD

Without any calculation, construct an isosceles right angled triangle equal in area to a rectangle 3 in by 1 3 in

5 If two circles cut, prove that the line joining their centres bisects the common chord at right angles

Word this enunciation so as to state a property of two isosceles triangles having a common base

6 Points X, Y, Z are taken on a circle so that YXZ is an acute angle and the minor arcs XY XZ are bisected at P and Q respectively The straight line PQ cuts XY, XZ in R and S Prove that $XR = XS$

No. 5

1. If a triangle has two equal sides, prove that the angles opposite those sides are equal.

Discuss the proposition—

If a triangle has two sides which are nearly equal, then the angles opposite those sides are also nearly equal.

2. In a triangle ACB the angle ACB is obtuse. Prove that the perpendicular from A on BC falls outside the triangle.

In what well-known proposition is this fact assumed without proof?

3. Prove geometrically that if a parallelogram and a triangle are on the same base and between the same parallels, the area of the parallelogram is twice that of the triangle.

Describe an equilateral triangle with side 1 in., and then construct a parallelogram equal to the triangle in area, and having the longer sides three times the length of the shorter sides.

4. Two parallel chords of a circle of radius 6·5 in. are respectively 12·6 in. and 12 in. long. If the centre of the circle is between them, what is the distance between the chords?

If the two chords had been at right angles, calculate the distance of their point of intersection from the centre.

5. What is the definition of *touching* circles? Prove that if two circles touch, their centres and point of contact are in a straight line.

If two circles touch at A, and any line through A meets one circle at P and the other at Q, prove that the radii drawn to P and Q are parallel.

6. Two circles touch at A; through A lines PAQ, XAY are drawn, meeting one circle at P and X, and the other at Q and Y. Prove that PX and QY are parallel.

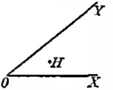
No. 6

1 Correct the following enunciations—

(a) Two triangles are congruent if they have two sides and an angle of one respectively equal to two sides, and an angle of the other

(b) Two triangles are congruent if they have the angles of the one equal to the angles of the other, each to each

2 Define a parallelogram, and from the definition prove that the diagonals of a parallelogram bisect one another



OX, OY represent straight portions of two railway lines, H represents a house which is known to be equi-distant from two stations, one on each railway line, and in the same straight line with them State how the positions of the railway stations may be found If the figure is drawn to the scale of 1 cm to the mile, what is the distance between the stations ?

3 Construct a triangle ABC of area 5 1 sq cm, having $AB = 3\ 4$ cm, $BC = 3\ 7$ cm

Measure the difference between the two possible values of the angle B

4 Divide a straight line AB into two parts at C so that the square on AC is equal to twice the square on BC, and give a geometrical proof that the construction is correct

5 If a circle can be inscribed in a quadrilateral $ABCD$, prove that $AB + CD = AD + BC$

Construct such a quadrilateral, being given that the radius of the inscribed circle is 5 cm angle A is $70°$, angle C is $80°$ $AB = 12$ cm Verify that $AB + CD = AD + BC$

6 An arc AB of a circle is bisected at D, and D is joined to the other extremity C of the diameter AC If CD cuts AB at E prove that rect DE DC = sq on DA

No. 7

1. Draw a line parallel to a given line AB through any given point P with ruler and compasses only. Prove your construction to be correct.

2. Define a rhombus, and from your definition prove that the diagonals of a rhombus bisect one another at right angles. Show that if a rhombus can be inscribed in a circle it must be a square.

3. In a triangle ABC the perpendicular from B on the bisector of the angle C meets AC (produced if necessary) at D. Prove that AD is equal to the difference between AC and BC.

4. If D is the middle point of the side BC of a triangle ABC, prove that—

$$AB^2 + AC^2 = 2BD^2 + 2AD^2.$$

Prove that, if the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the diagonals, the quadrilateral must be a parallelogram.

5. Prove that the greatest line that can be drawn from a point inside a circle to the circumference is that which passes through the centre.

A and B are two points within a circle, not on the same diameter. Find the point P on the circumference such that $AP^2 + BP^2$ is as great as possible. Prove your construction to be correct.

6. If at a point a tangent and a chord are drawn, prove that the angles between the tangent and chord are equal to the angles in the alternate segments of the circle.

At a point C on a circle a tangent is drawn which meets any chord AB, produced, at D. Perpendiculars DE and DF are let fall on CB and AC, produced. Prove that EF is at right angles to AD.

No 8

1 The side BC of a triangle ABC is bisected at D, and AD is produced to E so that $AE = 2AD$ Prove that $ABEC$ is a parallelogram

Construct a triangle ABC having $AB = 2$ in, $AC = 3$ in, and AD, the median bisecting BC, $= 2\ 3$ in Prove the truth of your construction

2 In the side BC of a triangle ABC any point P is taken, show how to draw a line PQ, cutting BA in Q, so that the triangle PBQ is equal in area to the triangle ABC

3 If a straight line is divided into two equal and also two unequal parts, prove geometrically that the rectangle contained by the unequal parts together with the square on the line between the points of section is equal to the square on half the line

State the equivalent algebraical formula

4 Prove fully that angles at the circumference of a circle standing on the same chord are either equal or supplementary

P is any point on the circumference of a circle of which AB is a fixed chord The bisectors of the angles PAB, PBA meet at I, find the locus of I

✓5 A line XY is drawn 6 8 cm from the centre of a given circle of radius 4 cm. Draw a circle of radius 3 5 cm. to touch XY and the given circle State the steps of your construction and prove your construction to be correct

6 Two straight lines AD, BC meet at O when produced Prove that, if rectangle $OA\ OD$ = rectangle $OB\ OC$, the angle ABD = the angle ACD

No. 9

1. The sides AB, CB of a triangle ABC are produced to P and Q respectively so that $BP = AB$ and $BQ = CB$. Prove that PQ is equal and parallel to AC. Give the enunciation of all the propositions used in the proofs.

2. Prove that parallelograms on the same base and between the same parallels are equal in area.

A point X is taken in the side AB of a parallelogram $ABCD$; prove that the triangle CXD equals the sum of the triangles BXD and AXD.

3. A straight line PO meets another straight line XY at O, and the angles POX, POY are bisected. From any point A on the bisector of POX a perpendicular AB is drawn to OP, and produced to meet the bisector of POY at C. Prove that the square on AO is equal to the rectangle $AB \cdot AC$.

4. In a triangle ABC, $AB = 3{\cdot}9$ cm., $BC = 4{\cdot}8$ cm., and the angle $BAC = 65°$; construct a rectangle equal in area to the triangle ABC, having one side of length $6{\cdot}5$ cm.

5. Give the construction, with proof, for describing a circle about a given triangle.

Two triangles have a side of one equal to a side of the other, and the angles opposite those sides equal; prove that the radii of their circumcircles are equal.

6. A circle is described having as diameter a radius OA of a circle with centre O. From A a straight line is drawn cutting the inner circle at P and the outer circle at Q. Prove that AQ is bisected at P.

Also if PR is at right angles to AO and is produced to meet the outer circle at S, prove that $AS^2 = 2AP^2$.

No 10

1 A picture is supported by a string 4 ft long which is attached to two rings in the top of the picture frame, 2 ft apart The string passes over a nail in the wall and the top of the picture frame is horizontal If the string is shortened by 1 ft, find, by drawing a figure to scale, by how much the picture will be raised

2 A transversal cuts two parallel lines at A and B The two interior angles at A are bisected and so are the two interior angles at B, the four bisectors forming a quadrilateral $ACBD$ Prove that (i) $ACBD$ is a rectangle, (ii) CD is parallel to the original parallel lines

3 If the square on one side of a triangle is equal to the sum of the other two sides, prove that the triangle is right angled

The middle points of the sides BC, CA, AB of a triangle are P, Q, R respectively, and AP, BQ, CR intersect at G If $AP = 24$, $BQ = 30$ $CR = 18$, prove that the angle PGC is a right angle

4 Show that four circles can be drawn so that each of them touches each of three given intersecting straight lines

5 Two lines AX, AY meet at an angle of $43°$, along AX two points B and C are taken so that $AB = 1·7$ in, and $AC = 2·5$ in Construct a circle that shall pass through B and C and have its centre 1 in from AY Show that there are two such circles

6 Any three points A, B, C are taken on the circumference of a circle, AD is drawn at right angles to AB to meet the circle at D, and CE is drawn at right angles to CA to meet the circle at E Prove that DE is equal to AB

No. 11

1. Prove that the angles at the base of an isosceles triangle are equal.

The base angles B and C of an isosceles triangle ABC are bisected by lines meeting at D. Prove that AD bisects the angle A.

2. In what kinds of parallelograms—

(i) Are the diagonals at right angles ?

(ii) Do the diagonals bisect the angles ?

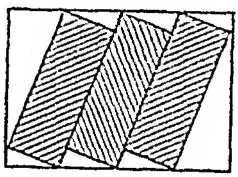
(iii) Are the diagonals equal ?

Lines are drawn joining the middle points of adjacent sides of a rhombus. Prove that the quadrilateral so formed is a rectangle.

3. Prove that triangles on the same base with their vertices on a line parallel to the base are equal in area.

Draw a triangle ABC having $AB = 4{\cdot}7$ cm., $BC = 3{\cdot}6$ cm., $CA = 5{\cdot}2$ cm.; and then construct a triangle equal to ABC in area and having one side 7 cm. long, and another 3 cm. long.

4. The diagram represents three books, too tall for a book-shelf, which just fit in the shelf when inclined at 20° to the vertical. If the books are each 9 in. tall and 1·5 in. thick, find, by drawing to a scale 1 cm. = 1 in., the length and height of the shelf.



5. Draw two circles of radius ·7 in. to touch a given circle of radius 1 in., and to pass through a given point ·5 in. from the centre. Measure the distance between the centres of the two circles.

6. If in two circles equal chords subtend equal angles at points on the circumference, prove that the circles are equal.

In a triangle ABC the sides AB, AC are equal, a point P on the other side of the base BC is such that the angles APB, APC are equal and AP is not perpendicular to BC. Prove that the triangles APB, APC have the same circumscribing circle.

No. 12

1 In a certain pentagon the five sides are equal and the bisectors of the interior angles meet in a point Prove that all the angles of the pentagon are equal to one another, and that a circle can be described to pass through the middle points of the sides of the pentagon

2 A rectangle *ABCD* has *AB* of length 3 5 in , and *BC* of length 2 in Construct geometrically a rectangle *ACEF* of equal area and prove the construction to be correct

3 By a geometrical construction prove that the rectangle contained by the sum and difference of two straight lines is equal to the difference of the squares described on them

Draw a figure to illustrate the algebraical identity—

$$(a+b)^2-(a-b)^2=4ab$$

4 Explain what you understand by "a proof by *reductio ad absurdum*"

Write out the enunciation and proof of any proposition in which this method is used

5 If two chords of a circle intersect within a circle prove that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other

Two lines *AB* *CD* intersect at *O*, so that *AO* *OB* = *CO* *OD*, prove that the sum of the angles *ACB* and *ADB* is two right angles

6 The sides *BA* and *CA* of a triangle *ABC* are produced to *D* and *E* respectively, making *AD* equal to *CA* and *AE* equal to *BA* The bisector of the angle *BAC* meets *BC* in *H*, *DE* in *K*, and the circle circumscribing the triangle *ABC* in *L* Prove that—

(i) *B*, *L*, *D*, and *K* are concyclic

(ii) Rectangle *AH* *AL* = rectangle *BA* *AC*

(iii) The square on *AH* = rectangle *BA* *AC* − rectangle *BH* *HC*

AUTHOR'S NOTE

THESE "Points Essential to Answers" contain skeleton solutions to all the riders in the Test Papers; they must be used sensibly or the student will lose rather than gain by their use. The student should make every effort to answer questions before consulting the solutions and must recollect that they are not model solutions but merely outlines. When the solution is understood, the student is advised to draw a fresh figure with different letters, and then to write out a fuller solution such as is required in an examination.

Riders frequently admit of several methods of solution. If a student finds that his solution differs from that given here, he is advised to revise it carefully, making sure that he has used all the data, has not inadvertently assumed the required result in the proof, and that he can give a reason for every step in the argument. If he is still satisfied with his proof, and, as is very probable, finds it shorter than the book solution, he will not be more pleased than will

W. E. PATERSON.

TEST PAPERS IN GEOMETRY
POINTS ESSENTIAL TO ANSWERS

No 1

1 Since all angles at $B = 360°$, angle $DBE = 120°$, and triangles ABC DBE are congruent.

3 105°

4 $AC^2 + BC^2 = AD^2 + DC^2 + BD^2 + DC^2$

$AB^2 \quad = AD^2 + BD^2 + 2AD \cdot DB$

but $AB^2 \quad = AC^2 + BC^2 \qquad CD^2 = AD \cdot DB$

5 When circle touches AY AX and AY are tangents and centre is on bisector of XAY During rolling centre moves along line parallel to XA at distance 3 cm Final distance from A = 6 5 cm

6. Acute angle A is in segment larger than semi-circle. A is outside semi circle Angles BPC BQC in semi-circle are right angles. BQ CP are two perpendiculars from vertices. Hence AX is the third perpendicular as the three are concurrent

No 2

1 By first part angles at B and C are either equal or supplementary Being angles of a triangle, they are not supplementary they are equal $AB = AC$

3 Draw AD and erect perpendicular at D on both sides. With centre A radius 6 2 draw circle cutting perpendicular on one side at C With centre A radius 5 7 draw circle cutting perpendicular on both sides Hence there are two non congruent triangles ABC

4 $AE^2 = AB^2 + BE^2 = BC^2 + BE^2 = 2BC \cdot BD + CE^2$

5 Join A to centre O and draw chord BC at right angles to OA Draw any chord PQ = 3 in With centre O and radius the perpendicular from O to PQ draw circle from A draw tangent to this circle and produce both ways cutting first circle at D and E Chords DE and PQ are equi distant from O and therefore are equal i e DE = 3 in.

6 Draw horizontal XY 6 ft from ground. On BC draw segment to contain an angle 20° the arc cuts XY at the two positions of E Distance = 12 2 ft

No 3

3 Draw triangle ABE having AB = 1 1 AE = 1 15 BE = 75 Then E is intersection of diagonals and AB is one side Obtuse angle = 113°

4 If O is mid point of AB $PA^2 + PB^2 = 2AO^2 + 2PO^2 \qquad PO$ = 1 5 in and locus is circle with centre O radius 1 5 in.

5 If I is centre triangles BIC CIB AIB make up triangle BIC $\frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$ = area $\qquad r$ = area − semi perimeter $\quad 65^2 - 56^2 = 121 \times 9 = 33^2$ triangle is right-angled Area = 28 × 33 diameter = 24 yd.

6 $DAC + DCA = PDA = PAD = PAB + BAD$ But $PAB = DCA$ $DAC = BAD$

No 4

1 Triangles AED BCF are congruent 1 side and 2 angles.

2 Flagstaff is on perpendicular bisector of AB and on bisector of angle BAC Distance = 504 yd.

3. Each joining line = half a diagonal and is parallel to diagonal. $\therefore$ figure is parallelogram. Each triangle at corner is $\frac{1}{4}$ of a triangle cut off by diagonal. $\therefore$ 4 triangles = half the parallelogram.

4. Make square = rect. 3 by 2·6 and draw diagonal.

6. Join PZ, QY

$$\begin{aligned} YRS &= QPY + PYX = YPZ + QPZ + PZX \\ &\qquad\qquad\quad\;\, = X + \tfrac{1}{2}Y + \tfrac{1}{2}Z \\ ZSR &= PQZ + QZX = YQZ + PQY + QYX \\ &\qquad\qquad\quad\;\, = X + \tfrac{1}{2}Z + \tfrac{1}{2}Y \end{aligned}$$

$\therefore XRS = XSR. \quad \therefore XR = XS.$

No. 5

1. If a triangle has two sides which are nearly equal, then the angles opposite are either nearly equal or nearly supplementary.

2. Assume perpendicular falls inside and use *reductio ad absurdum*.

3. Bisect base BC of triangle at D. With centre B, radius 1·5, describe circle to cut parallel through A to BC at E. Complete parallelogram $BEFD$.

4. 4·1 in. Distance from centre

$= \sqrt{\text{sum of square of distances of chords from centre}} = \sqrt{8{\cdot}81} = 2{\cdot}97.$

5. Two isosceles triangles have angles at base equal. $\therefore$ angles at vertices equal. $\therefore$ radii parallel.

6. Use angle between tangent and chord = angle in alternate segment.

No. 6

2. Produce OH to K so that $OH = HK$. Through K draw parallels to OX and OY meeting OX at A, OY at B. Then A and B are the stations = 5·2 miles.

3. Draw XY parallel to AB at distance 3 cm. With centre B, radius 3·7, describe circle. This cuts XY in two places. $\therefore$ two solutions. Difference = 71° 40′.

5. Let two tangents from A be of length x, from By, from Cz, from Dw. Then $AB + CD = x + y + z + w$ and $AD + BC = x + y + z + w$.

Draw circle centre O, radius 5 cm., and any radius OP. Make angle $POQ = 110°$. Draw tangents at P and Q to meet at A. Produce AP to B so that $AB = 12$ cm. Draw BR the other tangent from B. Make $ROS = 100°$. At S draw tangent to meet AQ at D and BR at C.

6. $DAE = DCB = DCA$. $\therefore DA$ is tangent to circle AEC. $\therefore DE \cdot DC = DA^2$.

No. 7

2. Opposite angles equal and also supplementary. $\therefore$ figure is a square.

3. If the perpendicular from B on bisector meets bisector at E, then triangles CED, BED are congruent and $CD = CB$.

4. Let O be mid-point of diagonal BD of the quadrilateral $ABCD$. Then $AB^2 + AD^2 + BC^2 + CD^2 = 2AO^2 + 2BO^2 + 2CO^2 + 2BO^2 = BD^2 + 2AO^2 + 2OC^2$. Hence $AC^2 = 2AO^2 + 2OC^2 = 4AP^2 + 4PO^2$, where P is mid-point of AC. $\therefore PO^2 = 0$, i.e. P and O coincide, i.e. diagonals bisect one another. $\therefore$ quadrilateral is a parallelogram.

5. Bisect AB at C; then $AP^2 + BP^2 = 2AC^2 + 2CP^2$ $\therefore AP^2 + BP^2$ is greatest when CP is greatest, i.e. when CP passes through the centre.

6. Since $DFC + DEC = 2$ right angles. $\therefore CEDF$ is cyclic. $\therefore AFE = CDE$. Also CD is a tangent. $\therefore FAB = DCE$. $\therefore FAB + AFE = CDE + DCE$ = 1 right angle.

TEST PAPERS IN GEOMETRY POINTS ESSENTIAL TO ANSWERS

No. 1

1. Since all angles at $B = 360°$, angle $DBE = 120°$, and triangles ABC DBE are congruent

3 105°.

4 $AC^2 + BC^2 = AD^2 + DC^2 + BD^2 + DC^2$

$AB^2 = AD^2 + BD^2 + 2AD \cdot DB$

but $AB^2 = AC^2 + BC^2 \quad \therefore CD^2 = AD \cdot DB$

5 When circle touches AY AX and AY are tangents and centre is on bisector of XAY During rolling, centre moves along line parallel to XA at distance 3 cm Final distance from $A = 6 5$ cm

6 Acute angle A is in segment larger than semi circle $\therefore$ A is outside semi circle. Angles BPC, BQC, in semi circle, are right angles $\therefore$ BQ, CP are two perpendiculars from vertices Hence AX is the third perpendicular, as the three are concurrent

No. 2

1 By first part, angles at B and C are either equal or supplementary Being angles of a triangle, they are not supplementary $\therefore$ they are equal $AB = AC$

3 Draw AD and erect perpendicular at D on both sides With centre A, radius 6 2, draw circle cutting perpendicular on one side at C With centre A, radius 6 7, draw circle cutting perpendicular on both sides Hence there are two non-congruent triangles ABC

4 $AE^2 = AB^2 + BE^2 = BC^2 + BE^2 = 2BC \cdot BE + CE^2$

5. Join A to centre O and draw chord BC at right angles to OA Draw any chord $PQ = 3$ in With centre O and radius the perpendicular from O to PQ, draw circle from A draw tangent to this circle and produce both ways cutting first circle at D and E Chords DE and PQ are equi distant from O and therefore are equal i e $DE = 3$ in.

6 Draw horizontal XY, 5 ft. from ground. On BC draw segment to contain an angle 20°, the arc cuts XY at the two positions of E Distance = 12 2 ft

No. 3

3. Draw triangle ABE having $AB = 1 1$, $AE = 1 15$, $BE = 75$ Then E is intersection of diagonals and AB is one side Obtuse angle = 113°

4 If O is mid point of AB, $PA^2 + PB^2 = 2AO^2 + 2PO^2$ $\therefore$ $PO = 1 5$ in and locus is circle with centre O radius 1 5 in

5 If I is centre, triangles BIC, CIB, AIB make up triangle BIC $\therefore$ $\frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr =$ area. $\therefore$ $r =$ area $-$ semi perimeter $65^2 - 56^2 = 121 \times 9 = 33^2$ $\therefore$ triangle is right angled. Area $= 23 \times 33$, diameter = 24 yd.

6 $DAC + DCA = PDA = PAD = PAB + BAD$ But $PAB = DCA$ $\therefore DAC = BAD$

No. 4

1 Triangles AED, BCF are congruent, 1 side and 2 angles.

2 Flagstaff is on perpendicular bisector of AB and on bisector of angle BAC Distance = 504 yd.

3. Each joining line = half a diagonal and is parallel to diagonal. $\therefore$ figure is parallelogram. Each triangle at corner is $\frac{1}{4}$ of a triangle cut off by diagonal. $\therefore$ 4 triangles = half the parallelogram.

4. Make square = rect. 3 by 2·6 and draw diagonal.

6. Join PZ, QY

$$\begin{aligned} YRS &= QPY + PYX = YPZ + QPZ + PZX \\ &= X + \tfrac{1}{2}Y + \tfrac{1}{2}Z \\ ZSR &= PQZ + QZX = YQZ + PQY + QYX \\ &= X + \tfrac{1}{2}Z + \tfrac{1}{2}Y \end{aligned}$$

$\therefore XRS = XSR$. $\therefore XR = XS$.

No. 5

1. If a triangle has two sides which are nearly equal, then the angles opposite are either nearly equal or nearly supplementary.

2. Assume perpendicular falls inside and use *reductio ad absurdum*.

3. Bisect base BC of triangle at D. With centre B, radius 1·5, describe circle to cut parallel through A to BC at E. Complete parallelogram $BEFD$.

4. 4·1 in. Distance from centre

$= \sqrt{\text{sum of square of distances of chords from centre}} = \sqrt{8{\cdot}81} = 2{\cdot}97$.

5. Two isosceles triangles have angles at base equal. $\therefore$ angles at vertices equal. $\therefore$ radii parallel.

6. Use angle between tangent and chord = angle in alternate segment.

No. 6

2. Produce OH to K so that $OH = HK$. Through K draw parallels to OX and OY meeting OX at A, OY at B. Then A and B are the stations = 5·2 miles.

3. Draw XY parallel to AB at distance 3 cm. With centre B, radius 3·7, describe circle. This cuts XY in two places. $\therefore$ two solutions. Difference = 71° 40′.

5. Let two tangents from A be of length x, from $B\,y$, from $C\,z$, from $D\,w$. Then $AB + CD = x + y + z + w$ and $AD + BC = x + y + z + w$.

Draw circle centre O, radius 5 cm., and any radius OP. Make angle $POQ = 110°$. Draw tangents at P and Q to meet at A. Produce AP to B so that $AB = 12$ cm. Draw BR the other tangent from B. Make $ROS = 100°$. At S draw tangent to meet AQ at D and BR at C.

6. $DAE = DCB = DCA$. $\therefore DA$ is tangent to circle AEC. $\therefore DE \cdot DC = DA^2$.

No. 7

2. Opposite angles equal and also supplementary. $\therefore$ figure is a square.

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4. Let O be mid-point of diagonal BD of the quadrilateral $ABCD$. Then $AB^2 + AD^2 + BC^2 + CD^2 = 2AO^2 + 2BO^2 + 2CO^2 + 2BO^2 = BD^2 + 2AO^2 + 2OC^2$. Hence $AC^2 = 2AO^2 + 2OC^2 = 4AP^2 + 4PO^2$, where P is mid-point of AC. $\therefore PO^2 = 0$, i.e. P and O coincide, i.e. diagonals bisect one another. $\therefore$ quadrilateral is a parallelogram.

5. Bisect AB at C; then $AP^2 + BP^2 = 2AC^2 + 2CP^2$ $\therefore AP^2 + BP^2$ is greatest when CP is greatest, i.e. when CP passes through the centre.

6. Since $DFC + DEC = 2$ right angles. $\therefore CEDF$ is cyclic. $\therefore AFE = CDE$. Also CD is a tangent. $\therefore FAB = DCE$. $\therefore FAB + AFE = CDE + DCE$ = 1 right angle.

No. 8

1 (i) Triangles ABD, EDC are congruent, 2 sides and included angle
(ii) First construct triangle ABE having $AB = 2$, $BE = 3$, $AE = 4\ 6$ Complete parallelogram $ABEC$, then ABC is required triangle

2 Draw CQ parallel to PA to meet BA at Q

4 If P is in major arc, angle APB is constant and acute, and angle
$$AIB = 90 + \tfrac{1}{2}APB$$
If P is in minor arc, angle APB is constant and obtuse and angle
$$AIB = 90 + \tfrac{1}{2}APB$$
$\therefore$ complete locus of I is the arcs of two segments, one on each side of AB, that on the major side containing an angle $90 + \frac{x}{2}$, that on the minor containing $180 - \frac{x}{2}$ where $x°$ is the acute angle APB

5 Centre of required circle is on line parallel to XY at distance 3 5 cm, and is also on circle centre O, radius 7 5

6 Because rectangles are equal, the points A, B, C, D are concyclic.

No. 9

2 Triangle BXD and triangle $AXD = \triangle ADB = \frac{1}{2}$ parallelogram = triangle CXD

3 Bisectors of adjacent angles are at right angles triangle AOC is right angled *with OB perpendicular to hypotenuse BC Hence, by proof of* Pythagoras, $AO^2 = AB \cdot AC$

4 Construct rectangle $ABPQ$ equal to triangle ABC, produce AB to D, making $AD = 6{\cdot}5$ Draw BF parallel to DQ, meeting AQ at F, then $ADEF$ is required rectangle

5 Angles at centres subtended by the equal sides are equal hence the radii are equal.

6 Angle APO is a right angle, being in semi circle OP bisects AQ. If AOB is diameter of large circle then ASB is a right angled triangle with SR perpendicular to hypotenuse AB $AS^2 = AB \cdot AR = 2AO \cdot AR = 2AP^2$

No. 10

1 61 of a foot = 7 3 in

2 Bisectors of adjacent angles are perpendicular angles at A and B are right angles Angles on same side of transversal together equal two right angles their halves are equal to one right angle angles C and D are right angles Again, diagonals of rectangle and bisect one another angle DCA = angle BAC = angle CAX where AX is the original parallel AX parallel to DC

3 Produce AP to H so that $GP = PH$ It is known by proof of concurrency of medians that $GH = \frac{2}{3}AP$ $CG = \frac{2}{3}CR$ and $CH = \frac{2}{3}BQ$ But $BQ^2 = AP^2 + CR^2$ $CH^2 = GH^2 + CG^2$ and CGH is a right angle

5 Centre must lie on right bisector of BC and also on one of the lines parallel to AY at distance 1 in. from it

6 AE and BD are diameters and bisect one another Hence $ABED$ is a parallelogram. $AB = DE$

No. 11

1 Halves of base angles are equal. $DB = DC$. $\therefore$ triangles ADB, ADC are congruent

2 Diagonals of rhombus are at right angles. Lines joining mid points of two sides of a triangle are parallel to the third side.

3. Draw CX parallel to BA; with centre B, radius 7, describe circle cutting CX at D. Then traingle BAD = triangle ABC. Through A draw AY parallel to BD, with centre D, radius 3, describe circle cutting AY at E. Triangle BDE = triangle BDA.

4. Length, 7·68 in.; height, 8·97 in.

5. With radius ·3 and centre as radius, describe circle, with point as centre and radius ·7, describe circle. Points of intersection are required centres. Distance is 0·52 of an inch.

6. Since A is on the right bisector of BC and PA bisects angle BPC. $\therefore$ $ABCP$ are concyclic. $\therefore$ triangles APB, APC have same circumcircle.

Or equal chords AB, AC subtend equal angles at P, $\therefore$ circles are equal. Angles ABP, ACP are not equal, $\therefore$ circles coincide.

No. 12

1. Let $ABCDE$ be pentagon, and let O be point where bisectors of angles meet. Then each adjacent pair of triangles are congruent, 2 sides and included angles. Hence all angles at bases are equal and angles of pentagon are equal. Also perpendiculars from O bisect sides of pentagon and are equal.

2. Draw through D a parallel to AC and complete the rectangle with AC as base.

5. For points A, B, C, D are concyclic.

6. Triangles BAC, EAD are congruent and AK bisects angle DAE; hence $AK = AH$.

(i) $\therefore$ angle BLK = angle BCA = angle BDK, and B, L, D, K are concyclic.

(ii) $\therefore$ rectangle $AH \,.\, AL$ = rectangle $AK \,.\, AL$ = rectangle $BA \,.\, AD$
= rectangle $BA \,.\, AC$.

but rectangle $AH \,.\, AL$ = rectangle $AH(AH + HL)$ = square on AH
+ rectangle $AH \,.\, HL$.

(iii) $\therefore$ square on AH = rectangle $BA \,.\, AC$ − rectangle $AH \,.\, HL$
= rectangle $BA \,.\, AC$ − rectangle $BH \,.\, HC$.

No. 13

1. Triangles EBG, GCK, KDM, MAE are congruent.
$\therefore EG = GK = KM = ME$.
Also angle EGB + angle CGK = angle EGB + angle GEB = 1 right angle.
$\therefore$ angle EGK is a right angle, similarly for the other angles.

2. Each triangle at corner $= \frac{1}{2} \times \frac{2}{3} \times \frac{1}{3} = \frac{1}{9}$ of the square.
$\therefore$ $EGKM = \frac{5}{9}$ of the square.

3. Let $AB = a$, $BC = b$, $CD = c$. Then
$AB \,.\, CD + AD \,.\, BC = ac + b(a + b + c) = (a + b)(b + c) = AC \,.\, BD$.

4. See Paper 5, No. 5, and use *reductio ad absurdum*.

5. The isosceles triangle is largest. Measurement = 1·89 sq. in.

6. $SQP = 90 - PQB = 90 - PAC = SRP$. $\therefore$ P, Q, R, S lie on a circle.
Also $CR \,.\, CS = CQ \,.\, CP = CA \,.\, CB$ = constant.

No. 14

2. Angle $C >$ angle B $\therefore C + \frac{1}{2}A > B + \frac{1}{2}A > \frac{1}{2}$ two right angles > right angle. $\therefore ADB = C + \frac{1}{2}A >$ a right angle.

3. Draw $AD = 4·8$, cut off $AE = 3·2$, at E make $DEC = 42°$, at D make $EDC = 68°$. Complete parallelogram $AECB$. Area = 4·4 sq. cm.

4. (i) Prove by *reductio ad absurdum*, viz., 2 angles of triangle = 2 right angles.

(ii) If circles cut in 3 points, 2 common chords both centres where right bisectors intersect circles would have equal radii, which is not true

5 Draw in same straight line $AB = 3\ 4$, $BC = 2\ 3$; from B draw any line $BD = 2\ 8$ Draw circle ADC cutting DB, produced at E Measure BE

6 Angle QPR = angle PAB (in alternate segment) $= 90 - PBA = ORB = PRQ$

No. 15

1 Let $ABCD$ be quadrilateral and E intersection of diagonals

From 4 big triangles ABC, BCD, CDA, DAB we have $AB + BC > AC$, etc, whence sum of sides > sum of diagonals

From 4 small triangles ABE, BCE CDE, DAE, we have $AE + EB > AB$, etc., whence twice sum of diagonals > sum of sides

2 Triangles BAC, DAC are congruent Hence triangles DAE, BAE are congruent

3 See Paper 81, No 3 Triangles HEO, HFO are equal in area.

4 Place the two triangles with supplementary angles adjacent, they are then seen to have equal altitudes.

Triangle AGE = triangle ABC, equal sides and included angles supplementary, etc

5 If small circle cuts OA at Q between O and P and R between P and A then OR is longest line from O to small circle the chord perpendicular to OR is shortest chord that touches small circle The diameters that touch small circle are the longest chords.

6 PQ is chord subtending constant angle, radius of circumcircle is constant, etc

No. 16

2 5 06 ft

3 Let QP produced meet BA produced at R Angle OQP = angle OBR Therefore $PRB = QOB = POA = 45°$

5 Angles BFC, BEC are right angles B, F, E, C are concyclic, etc

6 Angle PAC = angle CBA (alternating segment) = angle CAB

AD is at right angles to CA and therefore bisects the angle supplementary to PAB

No. 17

2 P is intersection of right bisector of AB and the bisector of angle B

3 Suppose $AD = a$, $BC = b$, and $a > b$ Draw CE perpendicular to AD $CD^2 = CE^2 + DE^2 = (a+b)^2 + (a-b)^2 = 2(a^2 + b^2) = 2(AD^2 + BC^2)$

4 Angle $BAQ = 180 - BAP = \frac{1}{2}(360 - 2BAP) = \frac{1}{2}(360 - BOP) = \frac{1}{2}$ reflex angle BOP

5 On radius OA as diameter describe circle, with centre B and radius BA describe circle cutting the last circle at C Join AC and produce to P, draw BQ parallel to AP OCA is right angle in semi circle $AP = 2PC$

AP, BQ are parallel chords, $PQ = AB = BC$, also angle BAP = angle QPA angle BCA = angle BAC = angle QPA BC is equal and parallel to PQ $PC = BQ$ and $AP = 2BQ$

6 Triangles DAF, EAF are congruent, two sides and included angle angle AEF = angle ADF = right angle Hence A E, F, D lie on a circle Produce HF right bisector of AB to meet DC at G Then angle $FEG = 60°$ and triangle EFG is half an equilateral triangle

$$\therefore DF = FE = 2GE = 2CD\left(1 - \frac{\sqrt{3}}{2}\right) = CD\,(2 - \sqrt{3})$$

No. 18

1. Triangles BAD, CAE are congruent, two sides and included angle.
2. Draw PQ parallel to XA meeting AY at Q. On QY cut off $QB = QA$. Join BP and produce to meet AX at C. Then BC is bisected at P. Or method of Paper 6, Question 2, can be used.
3. Join BE. Triangle $BEF = \frac{1}{2}$ triangle ADE, triangle $BEC = \frac{1}{2}$ parallelogram $ABCE$. $\therefore$ triangle $BFC = \frac{1}{2}$ trapezium $ABCD$.
4. Both angles are equal to AED.
5. Let P be point of contact of third tangent. CO bisects angle ACD and DO bisects angle BDC, and these angles are supplementary. $\therefore COD$ is a right angle. $\therefore OP^2 = CP \cdot PD$ (Paper 1, Question 4) $= AC \cdot BD$.
6. $BQ^2 = BP \cdot BA = AP^2$. $\therefore BQ = BR = AP$. $\therefore AR = BP$.

No. 19

1. Triangles ADB, FEB are congruent. $\therefore$ angle $ABD =$ angle FBE. $\therefore$ angles $ABD + FBD =$ angles $FBE + FBD = 2$ right angles, etc.
2. Angles EDB, AEB are obtuse. Hence $DE < BE$ (triangle BDE) $< AB$ (triangle ABE). Place triangles so that right angles coincide, then A falls between F and D; hence B must fall beyond E. $\therefore$ angle ABC is less than angle DEF.
3. Measurement $= 6\cdot7$ sq. cm.
4. If AD is median, and AE the perpendicular on BC, then $AB^2 - AC^2 = BE^2 - EC^2 = (BE + EC)((BE - EC) = 2BC \cdot DE$.
5. Centres of both arcs lie on the perpendicular to PQ at Z. $XY = 7\cdot15$.
6. Draw right angled triangle having hypotenuse PQ, and $PR = 1$ in. Required line is parallel to QR at distance 2 in., on side remote from P.

No. 20

1. $x = 2y$. $\therefore y = 36$, $x = 72$. $AD = DB > DC$.
2. Let fall AD perpendicular to XY, produce to E, making $AD = DE$ Join BE cutting XY at C.
3. Draw DE perpendicular to AC and therefore parallel to BC. Then $CE = \frac{1}{4}CA$, etc.
4. IB, EB bisect adjacent angle. $\therefore IBE$ is a right angle; so also is ICE.
5 Triangles ACD, BCE are congruent, two sides and included angle. $\therefore$ angle $CBE =$ angle DAC. Hence, angle $BFD =$ angle $ACB =$ angle ABC. $\therefore$ Circle BDF touches AB.

No. 21

2. Locus is a line XY parallel to base AB at distance 4 cm., such that $AXYB$ is a rectangle. Triangle of minimum perimeter is the isosceles triangle CAB, with AB as base. Produce BY to D, making $YD = BY$. Take any point P in XY. Then $CB = CD$ and $PB = PD$. Now $AP + PD > AD$. $\therefore AP + PB > AC + CB$.
3. Draw $AB = 4\cdot3$; mark off $AP = 3$. $PQ = 1$. On AQ as diameter draw semi-circle cutting PR (perpendicular to AQ) at R. On AB as diameter draw semi-circle, through R draw parallel to AB meeting this semi-circle at S. Draw SC perpendicular to AB. Then

$$AC \cdot CB = CS^2 = PR^2 = AP \cdot PQ = 3 \text{ sq. cm.}$$

4. Radius $= 3\cdot3$ cm.

5 If B and E are on same side of AD, then angle ABD = angle AED If B and E are not on same side of AD, then angle AED + angle $ACD = 180°$

6 Bisect AB at C, A and B being the centres of the circles. Draw QPR perpendicular to PC For proof, let fall perpendicular on QR from A and B

No. 22

1 Distance 95 1 yd.

2 Angles at point $= 5 \times 180° - 10$ angles at bases

$= 5 \times 180°$ – twice exterior angles of pentagon

$= 5 \times 180° - 2 \times 4$ right angles $= 180°$

3 Diagonal AC cuts BD at H and EF at G Since F and E are mid points of CD and CE, FE is parallel to BD and bisects HC ∴ $AG = 3GC$, and triangle $AFG = 3$ triangle CFG ∴ triangle $AFE = 3$ triangle CFE Also triangle $CFE = \frac{1}{4}$ triangle $CBD = \frac{1}{8}$ parallelogram triangle AFE + triangle $CFE = \frac{1}{2}$ parallelogram.

4 *Draw any chord of length 1 9 and mid point Q Draw concentric circle* with radius OQ From P draw tangent to this circle cutting outer circle at A and B

5 Produce AB, joining the two points to meet line at P Determine PQ the side of square equal to rectangle $PA \cdot PB$, and along line mark off PC equal to PQ Circle ABC is required circle, there are two solutions, as PC can be marked off in either direction

6. Reverse the proof for making isosceles triangle with each of the angles at base double of the angle at the vertex

No. 23

1 Draw line AB and erect equilateral triangle BAC, draw BAD perpendicular to AB Bisect angle CAD

3 $1\,7^2 - 1\,3^2 = 3 \times 4 = 1\,2 < 1\,21$ $\quad 1\,7^2 < 1\,3^2 + 1\,1^2$

triangle is acute angled and circumcentre is inside

4 (i) Bisect arc AB at C, then angle $AOB = 2$ angle $AOC = 2$ angle POQ

(ii) Chord AC = chord BC ∴ chord $AC > \frac{1}{2}$ chord $AB >$ chord PQ

∴ angle AOC is not equal to angle POQ Hence angle AOB is not double angle POQ

5 Since $ABD = ACD$, quadrilateral is cyclic ∴ $BE \cdot ED = AE \cdot EC$ ∴ $ED = 3$ Draw $AC = 3\,5$ cm and cut off $AE = 2$ cm On AE describe segment to contain an angle 100°, in it draw chord $EB = 1$ cm Produce BE to D, making $ED = 3$ cm

6 (i) $AP = AQ$ $BP = BR$, $CQ = CR$ etc

(ii) $AX = AB - BX = AB - BZ$ $AY = AC - CX$, $AX = AY$

$$2AX = AB + AC - BC$$

(iii) $AC + CR = AC + CQ = AQ = AP = AB + BR$

$$AC - AB = BR - CR$$

$$AC - CZ = AC - CY = AY = AX = AB - BZ$$

$$AC - AB = CZ - BZ$$

$$2(AC - CB) = BR - BZ + CZ - CR = 2ZR$$

No. 24

1 Angle ABC = angle $ACB = 75°$, etc

2 Through P draw a parallel to either diagonal of parallelogram $ABCD$

3 Let p in, q in, be lengths of perpendiculars from opposite vertices on diagonal of length 8 in Then area $= \frac{1}{2} \times 8 \times (p + q)$ ∴ $p + q = 6$ = other diagonal. ∴ the perpendiculars coincide with diagonals, i e diagonals are at *right angles Second part follows by Pythagoras*

4. Describe concentric circles with radii $(4+3)$, $(4-3)$, and $(5+3)$, $(5-3)$ respectively. The points of intersection give possible centres for the third circle.

5. Quadrilateral $PHAL$, $PKBL$, are each cyclic. Angle $PLH = PAH$ (same segment) $= PBA$ (alternate segment) $= PKL$ (same segment), etc.

6. Triangle is right angled. $\sqrt{5} = \sqrt{2^2 + 1^2}$. $\sqrt{7}$ = side of square = rectangle 7 by 1. A neat method is: Draw equilateral triangle ABC with side 2 cm. Let fall AD perpendicular to BC; produce DC to E, making $DE = 2$ cm. With D as centre, radius DA, describe circle cutting AD produced at F; with A as centre, radius AE, describe arc cutting previous circle at G. Triangle AFG is required triangle.

No. 25

1. $PR = 4{\cdot}8$ cm., $QR = 3{\cdot}6$ cm.

2. By book-work, DE is parallel to BA and DF to CA. Hence $AEDF$ is a parallelogram and diagonal AD is bisected by diagonal FE. Also triangle DEF = each of the triangles AFE, BDF, $CED = \frac{1}{4}$ of triangle ABC.

3. See Paper 19, Question 4.

Or, if $AB > BC$, ADB is obtuse and $AB^2 = BD^2 + AD^2 + 2BD \,.\, DX$
ADC is acute and $AC^2 = CD^2 + AD^2 - 2CD \,.\, DX$
$\therefore AB^2 - AC^2 = 2BC \,.\, DX$.

4. Let fall CD perpendicular to AB, then rectangle $AB \,.\, AD$ = square on AC.

5. Make $AOB = 40°$, $ABC = 95°$, $CD = CB$. Angle $BAD = 2 \angle BAC = 130°$.

6. Let circles intersect at AB; the common chord AB produced is locus, for if P is any point on that line, rectangle $PA \,.\, PB$ = square on either tangent. If circles do not intersect, let A and B be centres and R, r the radii.

Let PQ be perpendicular to AB. Tangents are equal. $\therefore PA^2 - R^2 = PB^2 - r^2$.
$\therefore PA^2 - PB^2 = R^2 - r^2$. $\therefore AQ^2 - QB^2 = R^2 - r^2$.
$\therefore (AQ + QB)(AQ - QB) = R^2 - r^2$.
$\therefore AQ - QB = (R^2 - r^2) \div AB$ = constant.
$\therefore Q$ is a fixed point, and locus of P is perpendicular to AB at Q.

No. 26

1. Three lengths must be measured.

2. Triangles ABP, ACQ are congruent (1 side and 2 corresponding angles). $\therefore AP = AQ$ and hence angle APQ = angle ACB. $\therefore PQ$ is parallel to CB.

3. Square AF = parallelogram $ACHL$. (Same base, AC.)
But angle $HCF = 90 - ACH = ACB$. $\therefore$ triangles ABC, FHC are congruent (1 side and 2 corresponding angles). $\therefore CH = CB = CD$. $\therefore$ parallelogram $ACHL$ = rectangle CK.

4. Angle XCE = supplement of $BCX = XAB = XCD$ (since $XB = XD$).

5. $DA = DB = DC$ (tangent from external point). $\therefore$ angle BAC is a right angle. PD bisects BDA, QD bisects CDA. $\therefore PDQ$ is a right angle.

6. Take any point C on circumference and draw tangent $CD = AB$; with centre O of circle as centre, and radius OD, draw circle to cut AB produced at P, etc.

No. 27

1. Angle AEC = angle ACE. $\therefore$ angle DEC = angle BCE.
Angle ADB = angle ABD. $\therefore$ angle BDE = angle DBC.
Hence angles DEC and BDE together equal half angles of quadrilateral $BDEC$. $\therefore CE$ and BD are parallel.

2. 85°, 110°, 95°, 70°.

3. $17{\cdot}8^2 - 16^2 = 33{\cdot}8 \times 1{\cdot}8 = 169 \times {\cdot}36 = 7{\cdot}8^2$. $\therefore$ triangle is right angled.
$\therefore$ Area $= \frac{1}{2} \times 16 \times 7{\cdot}8 = 62{\cdot}4$ sq. in.

4 Use first part twice, double, add

5. Draw perpendicular at B Join BC and produce to cut circle at E, join EO and produce to cut perpendicular at F F is centre of one of required circles Join BD to cut circle at O, produce OG to meet perpendicular at H H is the other centre

6 Let PEQ be tangent to circle ABE

Then angle $PED = BEQ = EAB$ (alternate segment) $= ECD$ ∴ PEQ is a tangent to circle CED

Let RES be common tangent to circles ADE, BCE

Then angle $ADE = AER$ (between chord and tangent) $= SEC = EBC$ (alternate segment) ∴ AD parallel to BC and $ABCD$ is a parallelogram

No. 28

1 $OA = OA'$, $OB = OB'$, $AB = A'B'$ angle AOB = angle $A'OB'$

2 AP equal and parallel to DQ ∴ PQ equal and parallel to AD and BC ∴ the two parts of intercepted parallelogram equal the two parts AQ, BQ of parallelogram $ABCD$

3 $BD^2 = 65^2 - 60^2 = 25^2$, $CD^2 = 156^2 - 60^2 = 144^2$, $BC = 169$ and $169^2 - 156^2 = 65^2$

5. Diameter is 10 cm Shortest chord is 8 cm

6 O describes an arc of a circle, centre C, radius CO, intercepted between perpendicular to AB at A and perpendicular at same distance on other side of BC A describes arc of circle, centre C, radius CA, so that angle ACA = angle OCO $BA = 20$ in

No 29

2 PH, RL, QK are parallel and $PP = RQ$ $HL = LK$

$OL = OH + HL$ and $OL = OK - KL$ $2OL = OH + OK$

Through R draw SRT parallel to OK meeting PH at S, KQ at T (either produced, if necessary) Triangles SPP, QRT are congruent, and $SP = TQ$ Hence $2RL = PH + QK$

3 Area $= 6\ 15$ sq cm

4 $AP^2 + BP^2 = 2AO^2 + 2PO^2$ PO is constant

5 Quadrilateral $DBCE$ is cyclic

6 Draw $BC = 1$ in, and perpendicular $CD = \frac{1}{2}$ in

Centre D, radius DC, draw arc cutting BD, produced at E

Centre B, radius BE, draw circle

Centre C, radius BE, draw circle, cutting previous circle at A

ABC is required triangle

Let F be point where B circle cuts CB produced. Join AF

$$\begin{aligned} \text{Rectangle } CB\ \ CF &= CB\ \ BF + CB^2 = CB\ \ BF + BD^2 - DC^2,\\ &= CB\ \ BF + (BD + DE)(BD - DC)\\ &= CB\ \ BF + BF(BD - DC) = BF(2DE + BD - DE)\\ &= BF^2 \end{aligned}$$

∴ $CB\ \ CF = CA^2$ CA touches circle ABF angle BAC = angle BFA, but angle ABC (at centre) $= 2$ angle BFA (at circumference) $ABC = 2BAC$ and $BAC = 36°$

No. 30

1 By using Pythagoras, it is seen that $DE > CB$ $CE > CB$ angle $CAE > CAB$ See also Paper 19, Question 2

2 DE is parallel to BA, equal to $\frac{1}{2}AB$ equal to $\frac{1}{2}AC$, DK is $\frac{1}{2}EC$, but DH is also $\frac{1}{2}EC$ $DH = DK$

But HK is parallel to DC and AD is perpendicular to BC AD is perpendicular to HK, but $DH = DK$ AD is the right bisector of HK $AB = AK$

3. First make parallelogram equal to square, with sides 2 in. and $2\frac{1}{2}$ in. Then make rhombus with side $2\frac{1}{2}$ in. equal to parallelogram.

4. Draw triangle ABC and about it describe circle. On AB describe segment to contain angle 140°, arc cutting AC at O. Produce AO to cut circle at D. $CD = 2{\cdot}2$ in.

5. Draw any radius OA and make $AOB = 160°$, let fall OC perpendicular to AB and draw a circle with centre O and radius OC. From P draw tangent to this circle, cutting former circle in QR. The minor segment cut off by QR contains 100°.

6. Angle DBK = angle DAC = angle EBD. $\therefore$ triangles DBK, DBE are congruent. $\therefore DK = DE$.

No. 31

1. Join EO, FO; triangles AOC, BOD are congruent. $\therefore AC = BD$. $\therefore AE = BF$ and triangles EOA, BOF are congruent. $\therefore$ angle EOA = angle BOF, etc. Or, $\therefore$ diagonals bisect one another, $ACBD$ is a parallelogram. $\therefore AE$ equal and parallel to BF, etc.

3. First make square equal to 20, i.e. 5 by 4 rectangle. Divide line 10 long so that rectangle contained by parts equals the square.

4. (i) Diagonals bisect. $\therefore$ quadrilateral is a parallelogram in a circle. $\therefore$ a rectangle. $\therefore$ diagonals are diameters.

(ii) CD is bisected at O. $\therefore$ triangles BOC, AOD are congruent and $AO = OB$.

5. Triangles ABC, DBC congruent. $\therefore$ angle BAC = angle BDC, etc.
Chord AB = chord CD. $\therefore$ arc AB = arc CD. $\therefore$ angle ACB = angle CAD. $\therefore AD$ and BC are parallel.

6. (i) Join QB. Angle $PQB = \frac{1}{2}ADB = 30°$, angle $QPB = ACB = 60°$. $\therefore QBR$ is 90° and QR is diameter.

(ii) Angle $SRB = \frac{1}{2}ADB = 30°$, $SBR = 90°$. $\therefore ASB = 120°$, $AOB = 60°$. $\therefore S$ is on circle ACB.

No. 32

1. 10·4 ft.

2. If $ABCDEF$ is hexagon, produce BA to meet EF at G.
Each angle of hexagon is 120°.
$\therefore$ angle $AGF = 120 - 60$. $\therefore$ 60. $\therefore AGF + ABC = 180°$, etc.

3. Fig. $RACB$ is a parallelogram. $\therefore RA = BC$, similarly $AQ = BC$.
Hence perpendicular from A on BC is perpendicular bisector of QR. $\therefore$ etc.

4. $CO = CB$. $\therefore$ angle OCA = 2 angle BOC; also angle $OAC = OBC = BOC$. $\therefore AOD$ = 3 times BOC.

5. Let R be point of contact with BC. $AP = AB + BR$, $AQ = AC + CR$. $\therefore AP + AQ$ = perimeter. $\therefore AP = s$.

6. Length = 8·48 ft.

No. 33

1. Triangles PAC, QAB are congruent. $\therefore$ angles PBX and QCX are equal. Triangles PBX, QCX are congruent. $\therefore BX = CX$. Triangles BAX, CAX are congruent, etc.

2. Produce AB to E, making $AE = CD = 4{\cdot}5$. Make $BC = 3{\cdot}6$, $EC = 3$; complete parallelogram $AECD$.

3. (i) Triangles BAE, DAG are congruent (2 sides and included angle). $\therefore$ rectangles are equal.

(ii) In triangle DAE, $DE^2 = DA^2 + AE^2 - 2DA \cdot AH$, and
$DE^2 \quad DA^2 + AE^2 - 2AE \cdot AL$, etc.

4 Parallelogram inscribed in circle opposite angles equal and supplementary a rectangle Parallelogram described about circle opposite sides equal and one pair of opposite sides — other pair of opposite sides (Paper 5 Quest on 5) all sides equal

5 Let AB be common chord then $PT^2 = PA \cdot PB = PQ \cdot PP$ PT touches circle QRT

6 $BC = 3\ 9$ $CA = 4\ 44$ $AB = 3\ 11$

No 34

1 Cut off AE from AB equal to AC Triangles ADE ADC are congruent and $DE = DC$

Angle $BED > ADE > ADC$ and $ADC > EBD$

$BED > EBD$ $BD > DE > DC$

2 Draw perpendiculars DBF and ECG to BC at B and C Draw two parallels to BC at distance 2 cm one meeting the perpendiculars at D and E the other at F and G On BC as diameter describe circle cutting DE at H and K and FG at L and M The parallels now read $DHKE$ $FLMG$ The complete locus is made up of the four finite lines DH KE FL MG

3 C is mid point of EA and CF parallel to AB F is mid point of BE Also $DC = AB$ $2CF$

4 Draw circle first in it place chord $BC = 2$ in etc There are two solutions

5 Prove PC is a tangent to the circle ACF Angle $PCA = ADE$ (parallel) $= ABC$ (same segment) etc

6 Quadrilateral $PQRS$ can be superposed on quadrilateral $XPQR$ PS XR

Also from triangle PAX QBR AX BR $XR = AB$

Construction, From AC cut off AE AB draw ES parallel to AB meeting BC at S etc

$^2QB^2 = QR^2$ $2QB = QR\sqrt{2}$ $AP + PQ + QB = AB$ $x(1 + \sqrt{2}) = a$

No 35

1 From triangle AEC DFB AE DF $AEFD$ is a parallelogram etc

2 Area = 4 13 sq in

3 Pentagon $= \frac{1}{2}[HF + LD + AH] = \frac{1}{2}[AB^2 + BC^2]$
$= \frac{1}{2}[AC^2 + 2BC^2 + 2AC \cdot CB] - \frac{1}{2}[AC^2 + CF^2] + AC \cdot CB$

5 Angle B 60 $CAD = ABD$ and CA is tangent to circle ABD etc

6 AD bisects angle BAC and so passes through I

Angle $DIC = IAC + ICA = \frac{1}{2}(A + C)$

Angle $DCI = DCB + ICB$ $DAB + ICB = \frac{1}{2}(A + C)$ etc

No 36

2 Triangles XAZ YBX ZCY are congruent XYZ is equilateral

Let P be mid point of YC Then triangle XBP is equilateral XY is perpendicular to BP

3 First make parallelogram on base 5 cm equal to rectangle with one side 6 cm etc

4 Indirect proof Take AD to bisect angle and meet circumference at D Then arc DB = arc DC chord DB = chord DC D is on perpendicular bisector of BC

5 DE is common chord of two circles having as diameters AB and CK etc

6 Distance of A from D the mid point of BC is $\frac{1}{2}AC$

No. 37

1. $BDC = 56°$.
2. DF is parallel to CA. $FA > PF$. $\therefore DQ > DP$.
3. Draw BQ parallel to PC.
5. On AB as diameter, describe semi-circle, in it place chord $BC = 1{\cdot}9$ in., etc.
6. Let common tangent at A meet PT at S.
Angle $RAT = APT + ATP$, but $ATP = TAS$ and $APT = QAS$.
$\therefore$ angle $RAT =$ angle QAT.

No. 38

2. BD and CE are two medians of triangle ACD.
3. Distance $= \frac{1}{2}\sqrt{7}$ miles $= 1{\cdot}32$ miles.
4. Angle is $66°$.
5. Angle is $45°$.
6. Equate two values for $PA^2 + PB^2 + PC^2$.
P lies on each circle.

No. 39

1. House is centre of circumscribed circle.
2. $67\frac{1}{2}°$ and $135°$.
3. Angle ACD is a right angle; also $PD = PB$.
Hence $AP^2 + PB^2 = AP^2 + PD^2 = AD^2 = AC^2 + CD^2$; etc.
4. Line joining any two of the centres and line joining the other two bisect adjacent angles and are therefore perpendicular.
5. $A' = 66°$, $B' = 59°$, $C' = 55°$.
6. Triangles ABP, ACQ are congruent. $\therefore APB = AQX$. $\therefore A, P, Q, X$ are on a circle. So also are A, B, C, X and AX is common chord of two circles, etc.

No. 40

1. Triangles BCF, AED are congruent. $\therefore BF = DE$. Hence AC bisects EF, etc.
2. If AD is median, triangles BDA, CDA are equal; so also are BDG and CDG. $\therefore$ remainder $AGB =$ remainder CGA.
3. Draw EF parallel to DA to meet BA produced at H; draw CK parallel to DB to meet AB produced at K. DHK is required triangle.
4. Triangles AOC, BOD are congruent (2 sides and included angle).
5. $BFEC$ are concyclic. $\therefore$ angle $AEF =$ angle B. Similarly $DEC =$ angle B. $\therefore DEF = 180 - 2B$.
But BC is diameter and P the centre of circle $BFEC$. $\therefore$ angle $FPC = 2B$.
$\therefore FPD + DEF = 180°$, i.e. D, E, F, P lie on circle.
6. OP and OQ are at right angles to AD and BC.
Angle $AOC = 2$ angle ABC and angle $BOD = 2$ angle BCD.
$\therefore AOC + BOD = 2$ right angles. $\therefore BOC + AOD = 2$ right angles.
$\therefore COQ + AOP = 1$ right angle $= AOP + PAO$. Hence $COQ = PAO$.
$\therefore$ triangles COQ, AOP are congruent, and $OP = CQ$.

No. 41

2. D is intersection of bisector of exterior angles at A and C.
3. Triangle $BCG =$ triangle AEB (equal bases, same vertex).
Triangle $GBF =$ triangle GDE (equal bases, same vertex)
$=$ triangle $DEC +$ triangle DCG.
$=$ triangle ADC, since triangle $DCA =$ triangle ADE.
$\therefore$ triangle $EFG = EBC + BCG + GBF =$ quadrilateral $ABCD$.

4 Loci are lines parallel to AB at distance 1 25 cm

On AB as diameter describe circle in it put chord $BC = 3$ 5 With centre A radius AC, describe circle cutting loci at P and Q etc

5 See Paper 40, Question 5 which shows $XYC = ZYA$

6 Suppose AC greater than AB Cut off $AD = AB$, then ABD, AQR are both equilateral

$AR \quad RC = PR \quad RS = PQ \quad RS + QR \quad RS$

$AR \quad RD = AQ \quad QB = PQ \quad QR + PQ \quad RS$

$AR(RC - RD) = QR(RS - PQ) \qquad AC - AB = QS - PR$

No. 42

2 Triangles ABF, CDF equal in area (equal bases between same parallels) etc

3 $AB^2 = AC^2 + BC^2 - 2BC \quad CD = AC^2 + BC \quad CD + BC \quad BD - 2BC\,CD$
$= AC^2 + BC \quad CD$

4 Right bisectors of AB and BC meet at P

5 Draw BC at B make $CBI = 20°$, draw a parallel to BC distant 1 in to meet BI at I Describe the inscribed circle with centre I From B and C draw tangents to meet at A

6 Squares of tangents from point of intersection equal same rectangle

No 43

1 Produce BP to cut AC at D

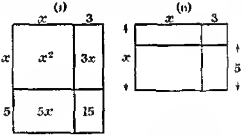
$BA + AC = BA + AD + DC > BD + DC > BP + PD + DC$
$> BP + PC$ Angle $BPC > PDC > BAC$

Neither converse is necessarily true

2 Angle CAD = angle BAD = angle ADC $\quad CD = CA$

If BD were parallel to AC figure would be parallelogram and $CD = AB$ Hence AB would equal AC, which is known not to be true

3



$(x - 5)(x + 3)$
$= x^2 + 3x - 5x - 15$

4 R mid point of hypotenuse $\quad OR = \frac{1}{2}PQ$ = constant

5 Angle ADC = angle BAD $\quad$ arc AC = arc BD $\quad$ chord AC = chord BD

Let common tangent QPR cut BC at Q

Angle $FEP = RPF$ (chord and tangent) $= BPQ = BAP$ $\quad EF$ is parallel to AD

Angle $EPC = 180° - EDC = DEF - 180 \quad FPD - BPD$

6 Median $AP = \frac{2}{3}AG = 3$ 6 Draw AD and erect perpendicular at D cutting circle centre A and radius 3 6 at P Draw perpendicular at P to PD cutting circle centre A and radius 2 at O the centre of the circumcircle etc

No. 44

1. Triangles ADF, BCG are congruent. $\therefore DF = CG$, and $EF = EG$.
2. Angle $EFG = 60° =$ angle FDC.
3. Make square equal to rectangle 7 by 3.
4. If D is mid-point of AB, $PA^2 + PB^2 = 2AD^2 + 2PD^2$. $\therefore$ point P is such that DP is greatest.
5. Diagonals bisect adjacent angles and are, therefore, at right angles. By congruent triangles, the diagonals bisect one another. Hence quadrilateral is a rhombus.
6. Angle PRQ is a right angle. $\therefore ABRP$ is a rectangle and $AP = BR$.

If O is mid-point of PQ, perpendicular from O on AB bisects AB and XY. $\therefore AX = BY$. $\therefore AX + AY = BY + AY = a$; also $BX \cdot BY = AX \cdot AY$.
But $BX \cdot BY = BR \cdot BQ = 1 \times b = b$. $\therefore AX \cdot AY = b$. $\therefore AX$. AY are roots of $x^2 - ax + b = 0$.

No. 45

1. Point is intersection of diagonals of the parallelogram.
2. $BC = 2{\cdot}67$ cm.
3. B, Y, X, C are concyclic. $\therefore AX \cdot AC = AY \cdot AB$.

$$BC^2 = BA^2 + AC^2 - 2BA \cdot AY = BA^2 - BA \cdot AY + AC^2 - CA \quad AX$$
$$= BA \cdot BY + CA \cdot CX.$$

4. Draw any transversal and bisect the four angles so formed. Line joining points of intersection of the bisectors would bisect the angle between original lines.
5. Produce BA to CD to meet at R; let S denote semi-perimeter of triangle PQR. Then $SB = SC = S$, $SA = SD = S - PQ$. $\therefore AB = CD = PQ$. See Paper 23, Question 6.
6. Angles ABR, ABS are right angles. $\therefore AP \cdot AR = AB^2 = AQ \cdot AS$, etc.

No. 46

1. Mirrors are perpendicular bisectors of line joining point to images and of line joining successive images. $\therefore OP = OP_1 = OP_2$, etc.
2. Through H draw PHQ parallel to AB and XHY parallel to AD, meeting AD in P, BC in Q, AB in X, DC in Y.

Triangle AHC + triangle AHD + triangle DHC = triangle ADC.
$\therefore 2AHC + 2AHD + 2DHC$ = parallelogram $ABCD$.
$\therefore 2AHC + 2AHD$ = parallelogram $ABCD$ − parallelogram $DPQC$.
= parallelogram $APQB$ = 2 triangle AHB.
$\therefore AHC = AHB - AHD$.

4. Place any chord $= 2{\cdot}9$ in circle, bisect it at C; draw concentric circle touching chord at C. From P draw PAB touching this circle.
5. Angle $OBA = OAB = ARB$ (alternate segment) $= AXO$ (parallel).
$\therefore X$ is on circle OAB, which passes through C, the centre of original circle.
$\therefore OXC = OAC$ = right angle, so X is mid-point of PQ.
6. Let O be centre of circle; on OC as diameter describe circle, cutting AB at P. Join CD, cutting circle at D and E.

No. 47

1. Triangle ABC, PQR congruent; triangles ADC, PSR congruent.
2. $EDB = 90° - EBD = 90° - ACB = BDF$.
4. $AD^2 = AB^2 + BD^2 = AC^2 + 2AC \cdot CB + 2BC^2$, etc.
5. Bisect line joining centres at C; join AC, draw PAQ at right angles to AC.

6 Draw isosceles triangle with angles at base double angle at vertex describe a circle about it Bisect angles at base

No. 48

1 Triangles BAQ, CAP are congruent

2 Let QA be perpendicular to YX, QB to XZ Through Y draw parallel to XZ, meeting QB produced at C. QC is perpendicular to YC, angle $CYZ = YZX = ZYX$. $QC = QA$ $QA - QC = BC =$ constant.

3 Construct triangle ABG, having $AQ = \frac{2}{3} \times 4\ 6$, $BG = \frac{2}{3} \times 3\ 5$ G is the intersector of medians AD and BE, etc

4 (i) Produce DC to H, making $CH = CD$. XH cuts AC at point P

(ii) Produce FD to K, making $DK = FD$, let fall KL perpendicular to AC produced, and produce KL to M, making $LM = LK$. XK cuts AC at point Q After striking AC the ball moves towards K, after striking CD it moves towards F

5 Semi perimeter $12\frac{1}{2}$ in Lengths are $12\frac{1}{2} - 9$, $12\frac{1}{2} - 10$ $12\frac{1}{2} - 6$ $AP = 12\frac{1}{2}$ (see Paper 23 Question 6)

6 Take P in arc AB Join HL, KL, required to prove $PLK + PLH = 2$ right angles

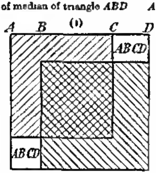
$PLK = PAK(PKAL$ cyclic$) = PBC(PABC$ cyclic$) = 180 - PLH(PLHB$ cyclic$)$

No. 49

1 34 7 ft

2 F is intersection of median of triangle ABD $AF = \frac{1}{3} AC$, etc

3 (i)



(ii) $(a + b + c)^2 + b^2 = (a + b)^2 + 2ac + 2bc + c^2 + b^2$

$= (a + b)^2 + (b + c)^2 + 2ac$

4 Let $ABCD$ be quadrilateral and H K the mid points of diagonals AC BD

$AD^2 + DC^2 = 2DH^2 + 2HC^2$ $AB^2 + BC^2 = 2CH^2 + 2BH^2$

$AB^2 + BC^2 + CD^2 + DA^2 = 4HC^2 + 4HK^2 + 4DK^2$

$= AC^2 + BD^2 + 4HK^2$

5 Radii to B and C make equal angles with AC and are parallel tangents are parallel

Angle CED = angle between tangent at C and chord CD = angle CDE (parallel) CE = arc CD

6 Draw circle, centre I radius 1 3 in Draw any radius IE and tangent at E With centre I, radius 2 5 in., strike arc cutting tangent at A, and draw AQ the other tangent Produce AE to P, making $EP = 4$, draw PH perpendicular to AP, meeting AI produced at H With centre H, radius HP, describe circle. Draw transverse common tangent meeting AQ at B, and AP at C.

No. 50

1. $AP + PB > AB$, etc. $\therefore AP + BP + CP >$ half sum of sides.
$AP + PB < AC + CB$ (Paper 43, Question 1), etc.
$\therefore AP + BP + CP <$ sum of sides.

2. Let bisectors of C and D of quadrilateral $ABCD$ meet at E, and of A and B meet at F.
Angle $E = 180 - \frac{1}{2}C - \frac{1}{2}D$, angle $F = 180 - \frac{1}{2}A - \frac{1}{2}B$. $\therefore E + F = 180°$.
If $E = 90°$, $\frac{1}{2}C + \frac{1}{2}D = 90$, $C + D = 180°$, and DA and CB are parallel.

3. Draw $AB = 3{\cdot}4$, produce to E, making $AE = 6$. With centre B, radius 4, and with centre E, radius $4{\cdot}3$, describe arcs cutting at C. Complete parallelogram $AECD$. Area $= 18{\cdot}5$ sq. cm.

4. Circle on OP as diameter. Longest chord is diameter, shortest is perpendicular to OP and is of length 8 cm.

5. Draw any circle passing through given points A and B, and cutting given circle at P and Q. Produce AB, PQ, to meet at R. From R draw tangent RT to first circle. Circle ABT is required circle.

6. If inscribed circle touches sides BC, CA, AB at D, E, F respectively, then circles with respective radii AF, BD, CE are required circles.
$A = 120°$, $B = 20°$, $C = 40°$.

No. 51

1. Triangles ACP, BCQ are congruent. $\therefore AP = BQ$, etc.

2. Similar to Paper 49, Question 4.

3. Angle $AEC = ABC + BAD = \frac{1}{2}AOC + \frac{1}{2}BOD = \frac{1}{2}AOB + \frac{1}{2}BOD = \frac{1}{2}AOD$.

4. Draw isosceles triangle with base 12, height 8. Inscribe circle in it. Radius is 3 in.

5. $AR = \frac{4}{5}AQ$ and $AQ = \frac{5}{9}AC$. $\therefore AR = \frac{4}{9}AC$. $\therefore AR : RC = 4 : 5$.

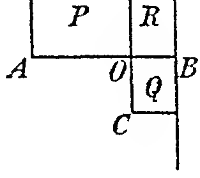
6. If $AB = 2$, $BC = \sqrt{3}$, $AC = 1$, $CD = 2 + \sqrt{3}$, $AD = \sqrt{8 + 4\sqrt{3}}$
$$= \sqrt{2}(\sqrt{3} + 1)$$
$$\therefore \sin 15° = \sin ADC = \frac{1}{\sqrt{2}\,(\sqrt{3} + 1)} = {\cdot}2588. \quad \tan 15° = \frac{1}{2 + \sqrt{3}}$$
$$= 2 - \sqrt{3} = {\cdot}2679$$

No. 52

1. PQ and RS each parallel to one diagonal, so are PS and QR. $\therefore PQRS$ is a parallelogram.

2. Right bisectors of AC and BC meet at V.

3. Angles ADC and ADB are each right angles.



5. Let $OA : OB = OC : OD$; place them as in the figure.
Then $P : R = OA : OB$
$Q : R = OC : OD$. $\therefore P = Q$.
Triangles POA, PAC are similar.
$\therefore PC : PA = PA : PO$.

6. $PN = a - r\cos x°$, $QN = r\sin x°$, PQ
$$= \sqrt{a^2 + r^2 - 2ar\cos x°}$$
$\therefore PQ$ is greatest when $\cos x°$ is least, i.e. $x = 180$, and least when $x = 0°$.

No 53

1 Let PQ produced meet AB at C Triangles APQ BPQ are congruent (3 s des) angle APC = angle BPC triangle APC BPC are congruent

2 From triangles DAB CBA $BD > AC$ From triangles DCB ADC angle $BCD >$ angle ADC

3 $AC = 6\ 5$ angle ADC is a right angle

Area $= \frac{1}{2} \times 5\ 6 \times 3\ 3 + \frac{1}{2} \times 6 \times 2\ 5 = 16\ 74$

4 A circle can be inscribed in the quadrilateral bisectors of angles meet at centre of circle

5 $2PQ = AB$ $2PR = AC$ PQ $PR = AB$ $AC = QX$ XR etc

6 $PQ = PT \cos P$ (since TQR is a right angle) $PT = TR \cot P$ (since PTR is a r ght angle)

$PQ = 9\ 6 \cot 63° 15 \cos 63° 15 = 2\ 18$

No 54

1 Four angles are needed.

2 Make $BC = 3\ 4$ angle $CBX = 4°°$ and cut off $BX = 1\ 6$ Draw per pendicular bisector of CX meeting BX produced at A $AC = 3\ 3$

3 Draw $AB = 3\ 2$ produce to C making $BC = 1\ 2$ At B erect perpen dicular meeting semi circle on AC as diameter at E From BA cut off $BF = 1\ 7$ join FE and draw EG perpendicular to EF meeting AC produced at G BG is other s de of rectangle

Or find the fourth proport onal to 1 7 3 2 1 2

4 (i) Perpendicular bisector of AB

(ii) Circle with centre C and radius = s de of square equal to CA CB

5 Draw isosceles triangle s des 3 in 3 in 2 in and make a square equal to it suppose side is x in Make another triangle similar to former with sides n ratio x 2 5

No 55

1 117° 108° 117° 99° 99°

2 Angle $DEA = EAB$ (parallel) $= DAE$ $DE = DA < AB < DC$

4 BI produced passes through D See Paper 35 Question 6

5 $BICE$ is cycl c since angles IBE CBE are right angles

triangles BID ECD are equiangular BI $CE = DB$ DE

6 Height 14 ft 4 in. Inclination, 15° 29

No 56

1 Angle XOY is a right angle

2 Draw CZ parallel to XO meeting OY at D Along DY mark off $DB = OD$ Join BC and produce to meet OX at A

3 Since rectangles are equal A B C D lie on a circle etc

4 See Paper 3 Quest on 5

73^2 $55^2 - 128 \times 18 = 64 \times 36 = 48^2$ triangle is right angled

$$\text{Area} = 48 \times 55 - 2 \quad \text{Hence radius} = \frac{48 \times 55}{176} = 15$$

5 (i) See Paper 52 Question 5

(ii) PC $CO = CA^2 - CA$ CB XC CY

Hence PC $CY = CX$ CO and included angles equal triangles PCX OCY are similar

(iii) PC $CO = XC$ CY P X O Y are on a circle

(iv) Angles OPX OPY subtend equal chords OX and OY

6 Distance from A 9 03 miles distance from road 4 70 miles. Direc tion 58° 38 W of S

No. 57

2. Draw through A, B, C, D lines parallel to the diagonals.
3. (i) Each parallelogram $= \frac{1}{2}$ whole parallelogram − halves of two small parallelograms.
   (ii) Triangles ABC, AYP, PKC are equiangular and similar.
5. See Paper 5, Question 5. Angle $P = \frac{1}{2}A + \frac{1}{2}B$, etc.
6. First make triangle BCD; on BD draw segment to contain 106°, and in it place $BA = 3$ in. Area 9·8 sq. in. Sides are $\frac{4}{5}$ the sides of $ABCD$.

No. 58

1. See Paper 20, Question 2.
2. See Paper 19, Question 4.
3. Draw XY parallel to AB at distance 2 in. On AB as diameter describe semi-circle, in it place chord $BP = 1{\cdot}25$. Produce AP to meet XY at D, etc.
4. At Q and R draw perpendicular to the tangents to meet at O. Prove $OQ = OR$.
5. Divide AB internally at X, externally at Y, in the ratio $2:1$. Circle on XY as diameter is required locus.
6. 69° 11′.

No. 59

1. If O is intersector of diagonals, triangles BOC, DOC are equal, also triangles BOE, COE.
2. Sides of isosceles triangle make equal angles with line through vertex parallel to base; therefore their sum is a minimum. (See Paper 20, Question 2.)
3. (i) Bisect line.
   (ii) All the rectangles have same perimeter.
4. (i) Rectangles have same area.
   (ii) Perimeter least when chord is least, i.e. when chord is perpendicular to line joining point to centre.
5. At A make $BAC = 22\frac{1}{2}°$, AC meeting perpendicular at B at C. Right bisector of AC meets AB at required point P.
6. Loci of vertices is circle on hypotenuse as diameter. $\therefore$ area greatest when height = radius of that circle. Hence maximum area is $\frac{1}{4}c^2$.

No. 60

1. $52\frac{1}{2}° = \frac{1}{2}(60° + 45°)$.
2. Join CE. Angles $KAE + BAC = 180°$.
   $\therefore$ angles $KEA + ACB = 90°$. $\therefore$ $KEC + BCE = 180$. $\therefore$ KE and BC are parallel.
3. At B erect perpendicular $BC = 2$ in. Draw CD perpendicular to AC, meeting AB at D. Mid-point of AD is required point P.
4. Locus is circle having as diameter the radius to A.
5. Let common tangent at C, the point of contact, meet AB at D. Then $AD = DC = DB$.
   Triangles AHD, BDK are equiangular (H and K being the centres). $\therefore$ $AH : BD = AD : BK$.
   $\therefore$ AD is mean proportional between radii and AB between diameters.
6. Part is 14 in. long, i.e. $(7 + 12) - (12 - 7)$.
   Maximum value of $OAP = \tan^{-1} \frac{7}{12} = 30° 15'$.

No. 61

1 Draw BH parallel to AD meeting DC at H, bisect BH at K Join EK, FK
Then EK and FK are both parallel to DC
Also $EK = \frac{1}{2}(AB + DH)$ and $FK = \frac{1}{2}(HC)$

2 Construct triangle CGH having $CG = \frac{2}{3} \times 1\ 9$, $GH = \frac{2}{3} \times 2\ 7$,
$$HC = \frac{2}{3} \times 3\ 4$$
Bisect GH at D, produce CD to B making $DB = CB$, etc

3 Angles ABC, ABD, subtended by equal chords, are equal

4 See Paper 23, Question 2 Inscribe circle in ABC, see Paper 50, Question 6

5 Angle $PDA = DBA$ (alternate segment) $= ADC$ (similar triangles since ADB is a right angle)
$PA\ \ AC = PD\ \ DC = PB\ \ BC$ (since DB, perpendicular to DA, bisects exterior angle)

6 $13^2 - 3\ 2^2 = 16\ 2 \times 9\ 8 = 3\ 24 \times 49 = 1\ 8^2 \times 7^2$ $\quad BD = 12\ 6$
$4^2 - 3\ 2^2 = 7\ 2 \times 8 = 1\ 44 \times 4 = 1\ 2^2 \times 2^2$ $\quad CD = 2\ 4$
$$BC = 15$$
$ABC = 14^\circ\ 15$, $ACB = 53^\circ\ 6$

No. 62

1 Cut off from DB, $DE = AB$ Draw EP parallel to DA

2 Draw quadrilateral $ABCD$ Draw CE, parallel to DB, meeting AB produced at E
Bisect AD at F, draw EG, parallel to BF, meeting AD at G BG bisects quadrilateral $ABCD$

3 Let circles with centres A, B, C, D touch at P, Q, R, S so that APB, BQC, CRD DSA are straight lines
Angle $APS = 90 - \frac{1}{2}A$, $BPQ = 90 - \frac{1}{2}B$ angle $SPQ = \frac{1}{2}A + \frac{1}{2}B$
Similarly, angle $SPQ = \frac{1}{2}C + \frac{1}{2}D$ $\quad SPQ + SRQ = \frac{1}{2}(A + B + C + D)$ $= 2$ right angles

4 Quadrilateral $AFDC$ is cyclic $\quad AK\ \ KD = CK\ \ KF$ etc

5 *See Paper 58 Question 5*

6 Along AX cut off AP such that $AP^2 = AB\ \ AC = 75 \times 91$ Angle BPC is maximum
Circle BPC touches AX at P Let O be its centre, and OD the bisector of BC

$$\text{Angle } CPB = \text{angle } BOD\,,\ \tan BOD = \frac{8}{\sqrt{75 \times 91}} \qquad CPB = 5^\circ\ 32$$

No. 63

1 Take any point P above AB and between A and B With centre P and radius PB describe circle Produce BP to meet circle at C AC is perpendicular to AB

2 Draw a transversal cutting AX and AY Bisect the four interior angles so formed Line joining the points of intersection of bisectors of angles is the required bisector

3 Bisect AB at C Draw AQ parallel to YB and BQ parallel to XP QC produced is the diagonal and that would pass through P

4 In AX take any point P and mark off $PQ = 2$ in draw PS parallel to $BY = 3$ cm, and complete parallelogram $PQPS$ Produce QS to meet BY at T Bisect QT at K, through K draw a line parallel to PR, this is the required line (Use fact that median of triangle bisects all lines parallel to base)

5. Join A and B to any point C on the arc AB. At A make angle equal to ABC and at B make angle equal to BAC.
Join P to A, cutting arc at D. Find PE the mean proportional to PA and PB. Circle with centre P, and radius PE, cuts arc at points of contact of tangents from P.
6. Distance $= 2\sqrt{5} = 4{\cdot}47$ in.

No. 64

2. Draw XY parallel to AB at distance 2 cm. With centres A and B, and radii each 3·5, draw circles meeting XY at D and E. DE is the complete locus.
3. See Paper 7, Question 4.
4. Describe circle to touch three sides, and then *reductio ad absurdum*.
5. Find a line, by geometrical construction, $= \sqrt{5}$. Increase sides in ratio $\sqrt{5} : 1$.
6. Construct triangle VAB with VC perpendicular to AB. From VA cut off $VP = 2$ cm., make angles $PVQ = 60°$, and $VPQ = 30°$. Produce VQ to R, making $VR = QP$. Join RA and draw QS parallel to RA, meeting VA at S. Draw ST perpendicular to VC. Section is a circle with centre T and radius TS.
Angle $AVB = 39° \ 14'$.

No. 65

1. P must be outside circle with centre A and radius 1·7 in.; inside a circle with centre B and radius 1·1 in.
2. Triangle ABC is half each of the three parallelograms formed.
XA, a median of XYZ, bisects BC, the other diagonal of parallelogram $ABXC$, and is a median of ABC.
4. Draw intersecting circles, one passing through A and C, the other through B and D. The chord of intersection cuts AD at O.
5. Bisect XZY by line ZO and make ZXO equal to CAP.
6. Height $= 20 \times \sin 56° \ 41' \div \sin 5° \ 9' = 186{\cdot}3$ ft.

No. 66

1. Triangles ABC, $A'BC'$ are equal in all respects.
Let AC, $A'C'$ meet at D, either or both being produced if necessary; and let BA', AC meet at E.
Angle $BAC = BA'C'$, $BEA = DEA'$. $\therefore ABA' = ADA'$.
2. Suppose $AB > AC$, then Y is in AC produced. Draw CZ parallel to AB, meeting XY at Z.
Then $CZY = AXD = CYZ$. $\therefore CZ = CY$. Triangles XBD, ZCD are congruent. $\therefore CZ = BX$, etc.
Note that area of triangle ABC is less than area of triangle PAQ where PQ is any line through D. The second part is solved by drawing BC so as to be bisected at D. See Paper 6, Question 2.
3. Locus is common chord produced.
5. Triangles BAD, PQS are similar (2 sides and incl. angle).
6. Suppose P is between A and C, consequently Q between E and B; then R is between D and C, and S is in DB produced. Prove $SQRP$ is cyclic.
Angle $SQD = SQB + BQD = BAE + BCP$
$= RPC + RCP = DRP$, etc.

No. 67

2 Draw $AB = 3\cdot8$, on AB as diameter describe a circle With centre B, radius 1 cut off BC from BA and BD on AB produced On CD draw equilateral triangle CDE Through E draw EF, meeting circle on AB at F Let fall FG perpendicular to AB Then $AO \cdot OB = 3$ sq in

3 See Paper 36 Question 5

4 See Paper 34, Question 6

5 (i) $\frac{AP}{PB} = \frac{AQ}{QB} \quad \frac{AO+OP}{OB-OP} = \frac{AO+OQ}{OQ-OB} \quad \frac{AO}{OP} = \frac{OQ}{AO}$

(componendo et dividendo)

(ii) $\frac{PB}{AP} = \frac{QB}{AQ} \quad \frac{AB-AP}{AP} = \frac{AQ-AB}{AQ} \quad \frac{AB}{AP} + \frac{AB}{AQ} = 2$

No. 68

1 16 right angles.

2 Let quadrilateral $ABCD$ be bisected by both AC and BD

Then triangle ADB = triangle ACB AB and CD are parallel, etc

3 The lines are the parallels to BC at distance equal to radius of given circle

4 See Paper 60, Question 3

5 Suppose P is mid point of side AB of a rectangle $ABCD$ and Q of side AD Bisect PQ at R Locus of A is circle centre R radius ½ in., Locus of C is a circle, centre R radius 1½ in

Produce PQ both ways to S and T so that $PS = PQ = QT$

Locus of B is circle on PS as diameter and of D circle on QT as diameter

6 From any point X on BA let fall XY perpendicular to BC and complete square $XYZW$

Join BW and produce to meet AC at Q Let fall QR perpendicular to BC, etc

From triangles APQ ABC, $\frac{x}{BC} = \frac{AP}{AB} = \frac{SD}{BD}$

From triangles PSB, ADB $\frac{x}{h} = \frac{BS}{BD} \qquad \frac{x}{BC} + \frac{x}{h} = 1$

No. 69

2 Perpendicular from C on BD is twice AB C is outside square beyond BD or beyond AE angle BAC or ABC is obtuse

If BAC is obtuse AE bisects BC if ABC is obtuse, BD cuts AC in a point of trisection

3 Length $= \sqrt{15} = 3\cdot87$ in

5. Angle $PYB = 180° - PAB = QAB = 180° - QXB$ etc

6 Radius of circle is constant in length locus is a circle centre O

No 70

1 See Paper 2 Question 1

Or produce AD to E, making $DE = AD$ Join EC

Triangles ADB CDE are congruent $AB = CE$, also angle $CAD = BAD = DEC$ $CE = AC$

2. Produce BA to E, making $AE = AC$. Then AD bisects EC at right angle. $\therefore DE = DC$, etc.

3. Draw $CD = 2{\cdot}5$. On CD as diameter describe semi-circle, and in it place chord $DE = 1$. Produce DE to B, making $DB = 3$; draw CA parallel to EB and equal to 2.

$AB^2 = CE^2 = 2{\cdot}5^2 - 1^2 = 3{\cdot}5 \times 1{\cdot}5$. $AB = 2{\cdot}29$ in. Area $= 5{\cdot}72$ sq. in.

4. $7{\cdot}4^2 - 7{\cdot}0 = 14{\cdot}4 \times {\cdot}4 = 2{\cdot}4^2$, etc.

5. See Paper 41, Question 6.

6. $OA = 41{\cdot}32$ ft., $OB = 37{\cdot}32$ ft., $OC = 41{\cdot}73$ ft., $OD = 37{\cdot}73$ ft.

No. 71

2. See Paper 33, Question 2.

4. If $BC = CA$, angle CAB = angle CBA = angle DBC. $\therefore DB$ is a tangent.

If P between B and D, then angle $DBC <$ angle between CB and tangent at B. $\therefore$ angle $CBA < CAB$, $\therefore AC < CB$.

5. Triangle ACD : triangle $ABD = 6^2 : 4{\cdot}5^2 = 16 : 9$ (similar triangles).

6. Height $= 134{\cdot}6$ ft. Angle of elevation $= 12° 48'$.

No. 72

1. Angles $A + B + C + D = 360$.

Angle $X = 180 - A - D$. Angle $Y = 180 - C - D$. $\therefore X + Y = B - D$.

Let OX cut BC at E.

$$\text{Angle } XOY = 180 - \frac{Y}{2} - OEB = 180 - \frac{Y}{2} - \frac{X}{2} - EBX = B - \frac{B - D}{2}$$
$$= \frac{B + D}{2}.$$

3. $15^2 - 13^2 > 7^2$. $\therefore$ triangle is obtuse angled. Perpendicular $= 6{\cdot}062$.

4. Describe circle, centre O, radius $3\frac{1}{2}$ in.; describe circle, centre P, radius 2 in. The two points of intersection of these circles are the possible centres. Distance $= 3{\cdot}92$ in.

6. Angle $= 2\cos^{-1}\dfrac{7{\cdot}2}{8} = 2\cos^{-1}{\cdot}9 = 51° 41'$.

No. 73

2. Quadrilateral $ABCD$ is such that triangles ABC, ADC are equilateral.

Draw BE, parallel to AC, meeting DC produced at E.

Mark off $DF = 2{\cdot}8$. Draw EG parallel to FA to meet DA produced at G. Triangle DFG is equal to $ABCD$.

3. (i) Circle having as diameter line joining centre of given circle to given point.

(ii) Diameter perpendicular to given line.

4. AP bisects angle BAC. $\therefore PX = PY$. Also $PB = PC$. $\therefore BX = CY$.

5. Let XAY be tangent at A to circle ABC.

Angle $XAB = ACB = AQP$. $\therefore XAY$ touches circle APQ.

For second part, draw tangent at O to circle POQ and similar proof follows.

6. XO bisects angle AXY, and YO bisects supplementary angle XYB, $\therefore XOY$ is a right angle. $\therefore$ angle BOY = angle OXA. $\therefore$ triangles OXA, OYB are similar.

Hence $BY : OB = OA : AX$. $\therefore BY = 2OA = 2BQ$. $\therefore YQ$ = radius.

No. 74

2 See Paper 62, end of Question 2

4 Centre is intersection of perpendicular bisectors of AB and CD

5 CB is tangent to circle A angle BCE = angle in alternate segment $= \frac{1}{2}$ angle CAB Similarly, $ACD = \frac{1}{2}CBA$

$\therefore ACD + BCE = \frac{1}{2}(CAB + CBA) = \frac{1}{2}$ a right angle $\therefore DCE = \frac{1}{2}$ a right angle

6 $R = \dfrac{a}{2 \sin A} = \dfrac{abc}{2bc \sin A} = \dfrac{abc}{4\Delta}$

For r, see Paper 3, Question 5

Let O be circumcentre, I the incentre of triangle ADC

Produce OI both ways to cut circumcircle at P and Q

Then $R^2 - OI^2 = PI \cdot IQ$

Produce AI to meet circle at D Join DO and produce to meet circle at E Join DC, DC is perpendicular to EC

Let fall IX perpendicular to AC

Triangles CDE, AIX are similar (equiangular)

$DE \cdot AI = DC \cdot IX$, i.e. $2Rr = AI \cdot DC$

But $DC = DI$ (Paper 35, Question 6)

Hence $R^2 - OI^2 = PI \cdot IQ = AI \cdot DI = AI \cdot DC = 2Rr$

No. 75

1 With radius 1 in., describe circles with centres A and B cutting at P With radius 1 in., describe circle with centre P cutting circle with centre B at Q

Circle with centre Q and radius 1 in., cuts circle with centre P at point C

2 Draw OPQ, parallel to DA and CB, cutting CD at P and AB at Q

Triangles $DOA + AOB + BOC$ = triangle DOC + parallelogram $ABCD$, but $DOA = \frac{1}{2}$ parallelogram DQ and $BOC = \frac{1}{2}$ parallelogram CQ

Triangles DOA and $BOC = \frac{1}{2}$ parallelogram $ABCD$ = triangle DAC

Triangle AOB = triangle DOC + triangle DAC

= triangle AOC + triangle AOD

3 DA is a median of triangle BDE, so is EC BF is the third median

4 (i) Make angle $= 360° \div 15 = 24$ at centre, this gives the side of figure

(ii) Let AB be side of inscribed equilateral triangle and AC a side of inscribed regular pentagon Bisect arc BC at D, then BD DC are two consecutive sides of the required figure

5 By similar triangles $OA : OD = OB : OC = 3 : 9$

Dividendo $OA : AD = OB : BC = 3 : 6$ $OA = 2$ cm, $OB = 2\frac{1}{2}$ cm

6 $\cos BCD = \frac{3}{4}$ $BCD = 41° 24'$

Area $= 6 \times 5 \sin BCD = 19.84$ sq cm

No. 76

3 $APQR$ is a parallelogram AR bisects PQ

5 Let P, Q, R be respective circumcentres of triangles ABC, ABD, ACD Then PQ is perpendicular to AB and PR is perpendicular to AC angle $QPR = 180 - BAC$

Angle AQD at centre $2ABC$ $AQR = ABC$

Similarly, $ARQ = ACB$ $QAR = BAC$, etc

6 Find fourth proportion to 3.1 cm, 2.3 cm, 1.8 cm value = 1.34

No. 77

2. See Paper 20, Question 2.
3. Draw any chord of length 5·2. Draw concentric circle touching this chord, and from P draw tangent to this circle.
4. Draw any circle with centre O passing through AB; at A draw AC a tangent of length 4 in. With centre O and radius OC, describe circle cutting AB produced in P.

Or bisect AB at D, at B erect perpendicular $BE = 4$. With centre D, radius DE, describe circle cutting AB produced at P.

5. Triangle ABC, XYZ are equiangular (angles in same segment).
6. Inscribed polygon is made up of n isosceles triangles with equal sides, equal to r and included angle $= \frac{360°}{n}$.

Described polygon is made up of n isosceles triangle with height r and angle between equal sides $\frac{360°}{n}$. Perimeter $= 2nr \tan \frac{180°}{n}$; area $= nr^2 \tan \frac{180°}{n}$.

No. 78

1. Angle $ACQ = 180° - APQ = BPQ = 180° - BDQ$, etc.
2. In right-angled triangle POQ, $OR = RP = RQ$. $\therefore$ locus of R is quadrant of a circle with centre O and radius $\frac{1}{2}$ of PQ.
3. For isosceles triangle, produce DC both ways, draw BH parallel to AC to meet DC at H, and EK parallel to AD to meet CD at K. AHK is isosceles triangle of equal area.
4. Let circles XBC, YCA meet at O.

Angle $BOC = 180° - X$; angle $COA = 180° - Y$. $\therefore$ angle $AOB = X + Y$.
$\therefore AOB + AZB = 180°$, and circle ABZ passes through O.

6. If $AB^2 = AQ \cdot BR$, then $AQ : AB = AB : BR$. $\therefore$ try to prove triangles ABQ, ABR to be similar.

By data, $RP \cdot RA = RB \cdot RC$. $\therefore ABCP$ is cyclic. Hence $PAQ = PBR$. $\therefore$ angle BAP = angle AQB, etc.

No. 79

1. Angles ICE, IBE are right angles. $\therefore B, I, C, E$ lie on a circle.

Also AIE lie on bisector of BAC, which bisects arc BC of circumcircle at D. Now $DB = DC = DI$ (see Paper 35, Question 6). $\therefore D$ is the centre of circle $BICE$.

2. Make triangle ABC. Produce BA to P so that $BA = AP$, and BC to Q so that $BC = CQ$; then PQ is parallel to AC. Make $BCD = 95$, D being on PQ.
3. $AX \cdot AY = AX^2 + BX \cdot XY = BX^2 + BX^2 + BX \cdot BY$.
4. Within angle XOY draw CH parallel to OX at distance equal to radius of given circle, and CK parallel to OY at distance equal to radius. These parallels are loci of centres of the circle. Distance between centres is bisected by R, the point of contact. $\therefore$ locus is quadrant of circle with centre C and radius equal to radius of given circles.
5. Use angle properties of cyclic quadrilaterals.
6. Triangles PXZ, PXY are equiangular.

No. 80

1 $D = F = \frac{8}{7}$ of right angle, $G = G = \frac{6}{7}$ of right angle

2 Draw BPQ parallel to YZX, cutting CZ at P and AX at Q, then $AQ = 2CP$ $AX + BY = AQ + QX + BY = 2CP + 2PZ = 2CZ$
In second case, $AX - BY = 2CZ$

3 $DE^2 = AE^2 + AD^2 = BC^2 + AD^2 = AB^2 + AC^2 + AD^2$
Similarly for CF^2

4 Draw OT common tangent at O
Angle $ROS = ROT - SOT = RQO - SPO$ (alternate segment) $= POQ$

5 PC $PD = PA \cdot PB = PT^2$, etc

6 On BC as diameter describe semi circle cutting CA at E In semi-circle place chord $CD = 2BE$

No. 81

1 By congruent right angled triangles, triangles ABC, PQR have equal altitudes etc

2 Let $ABCD$ be the parallelogram E the intersection of the diagonals, AP, BQ, CR, DS, EO the respective depths
By Question 2, Paper 80, $AP + CR = 2EO$ and $BQ + DS = 2EO$

3 $PMBK$ = triangle ABC − triangle AMP − triangle PKC
= triangle ADC − triangle AHP − triangle PLC = $PLDM$.
Make rectangle $PLDH$ having $HP = 3$ $PL = 1$, produce HP to K, making $PH = 2$ 3 Complete figure with lettering of first part of this question, then $PMBK$ is required rectangle

4 In circle A place chord = 5 cm and draw concentric circle touching chord
In circle B place chord = 3 cm, and draw concentric circle touching chord.
$PQRS$ is a common tangent to these concentric circles

5 See Paper 48, Question 6

6 Length $= \sqrt{20^2 + 15^2 + 12^2} = 27\ 7$ ft Angle $= \tan^{-1} 48 = 25° 38'$

No. 82

1 Draw any line XCY through C From A draw line AX, making $AXC = 60°$, from B draw BY, making $BYC = 60°$ Produce YB, XA to meet at Z

2 Let bisectors of angles A and B of parallelogram $ABCD$ meet at P Then $A + B = 2$ right angles APB is a right angle, etc

3 Draw HK parallel to PQ at distance 2, cutting OS at L
Draw XY parallel to PQ at distance 2, cutting OR at Q
Required locus is made up of the bisectors of the angles HLS, KLS, XZR, YZR

4 Divide AB at C so that rectangle AB $BC = AC^2$

5 $AK = \dfrac{AE}{\cos(90° - C)} = \dfrac{C \cos A}{\sin C} = \dfrac{a \cos A}{\sin A} = a \cot A$
Since PQ is of constant length angle PAQ is constant AK is constant

6 D is on perpendicular bisector of BC Prove triangles ABD, ACE to be equiangular

No. 83

1. See Paper 33, Question 1.
2. Produce BO to meet AC at D; then $BOC > BDC > BAC$.
Angle may be outside triangle and yet $> BAC$.
4. Opposite angles equal and supplementary.
Mid-points are corners of a parallelogram, sides parallel to diagonals, which must be a rectangle. ∴ diagonals are at right angles.
5. Triangles ABC, BCD are equiangular.
6. Shadow $= 38{\cdot}60$ ft.
Sun is due south. Shadow of wall is parallelogram, length 90 ft., height 7 tan 40 sin 45. Area $= 374$ sq. ft.

No. 84

2. Let fall AE, BF perpendicular to CD (AB being less than DC). Right-angled triangles ADE, BCF are congruent. ∴ angle $ADE =$ angle BCD. Hence triangles BCD, ADC are congruent. ∴ angle $DAC =$ angle DBC, etc.
3. ∵ AB is divided at P, $AB^2 + BP^2 = AP^2 + 2AB\,.\,BP$
$= 2BP^2 + 2AB\,.\,BP$.
∴ $AP^2 = 2BP^2$ and construction is same as Paper 59, Question 5.
4. Draw KXL not parallel to line of centres. Bisect KX at M, XL at N; then MN is $\frac{1}{2}KL$ and is less than line joining centres. If PXQ is parallel to line of centres, then $\frac{1}{2}PQ =$ line joining centres.
5. On side of PQ remote from A describe segment containing angles equal $90 + \frac{1}{2}PAQ$; draw bisector of PAQ to meet arc of segment at I. I is centre of inscribed circle, etc.
6. $AOXY$ is cyclic. ∴ $OYX = OAX$.
$BOZX$ is cyclic. ∴ $OZX = OBA$, but $OBA = OAX$, etc.

No. 85

1 Draw triangle ECB having $EC = 5{\cdot}6 - 3{\cdot}7$, $CB = 2{\cdot}5$, angle $CEB = 70°$. Produce CE to D, making $CD = 5{\cdot}6$. Complete parallelogram $DEBA$.
Area $= 10{\cdot}5$ sq. cm.
2. Their altitudes are equal, by congruent right-angled triangles.
4. See Paper 23, Question 6 (ii).
5. Suppose D were outside circle ABC, then angles ABC, ADC (opposite equal chords) would be equal, which is impossible.
6. $AI : ID = AB : BD = AC : CD = AB + AC : BC$.
Or produce BA to E, making $AE = AC$. Then angle $BAC = 2$ angle AEC ∴ AD is parallel to EC.
Hence $AI : ID = AB : BD = EB : BC = AB + AC : BC$.

No. 86

1. Let fall AD perpendicular to XY and produce to E, making $DE = DA$ EB produced cuts XY at required point C.
Or, if AB cuts XY at D, draw locus of point P such that $AP : PB = AD : DB$. Locus cuts XY at C.
2. Each side of quadrilateral $= \frac{1}{2}$ diagonal of rectangle.
Each triangle cut off at corner $= \frac{1}{8}$ of rectangle.
3. Triangles PAC, BAQ are congruent. ∴ angle $RCA =$ angle RQA, etc.

4. *See Paper 19, Question 4.*
Projection of medians of triangles ABC, ABD, which bisect AB, are equal and on same side of mid point of BC

5 See Paper 3, Question 5

6 Triangles PCA, OCA are equiangular
$PC \cdot CO = AC^2 = XO\ \ OY$ $\quad P, X, O, Y$ lie on a circle
$\therefore XPO = XYO = OXY = OPY.$

No. 87

1 *Triangles DAE, CBE are congruent.* $\therefore$ triangles DFE, CFE are congruent

2 Draw through B and D parallels to AC, bisect AC at P and let perpendicular bisector cut parallel through B at Q and that through D at R Produce PQ to V, making $QV = PR$ $\quad VAB$ is required isosceles triangle

3 Radius of sphere = radius of circle inscribed in triangle of sides 13, 13, 10

Radius $= \frac{60}{18} = 3\frac{1}{3}$ in.

5 Bisect PB at X, draw XY parallel to BC, meeting AC at Y. Join PY and produce to meet BC produced at Q

6 (i) *Draw isosceles triangle VAB having $VAB = VBA = 70°$* On VC, the perpendicular from V to AB, describe a semi circle and in it put $CD = CB$ VCD is required angle

(ii) Angle is $\cos^{-1}(\cot 70°) = 68° 39$

No. 88

1 *$AC = BC = 7\ 4$ cm*

3 Triangle DOC is $\frac{1}{2}$ an equilateral triangle $\quad OC = 2OD$ $\quad$ But $OA = OC$ (O is on bisector of angle B, which is perpendicular bisector of AC) $\quad OA = 2OD = 2OE$, *etc.*

5 Triangle ABE, CBD are congruent (2 sides and included angle)
$BAE = BCD$ $\quad$. AE produced passes through D

6 Height of triangle $ABC = 5 \tan 55°$ $\quad AC = BC = 7\ 414.$

No. 89

1 Triangles ABC, DBC are congruent angle DBC = angle ACB etc.

2 Let O be mid point of BC $\quad$ Angle BPC is a right angle $\quad BC = 2OP$ But P is equidistant from AB, BC, and CD $\quad OP$ is parallel to BA
$BC = 12$ in

3 Side of square = 1 65 sq in

4 From O the centre of circle, let fall a perpendicular to line through Q perpendicular to PQ, perpendicular from O cuts circle at R and S $\quad$ Join RQ, cutting circle at T, and SQ cutting circle at V $\quad OT$ cuts PQ at X and VO cuts PQ at Y $\quad$ Circles with centres X and Y satisfy conditions.

5 Triangles BOC, EOC are congruent $\quad OBC = OEC$
Triangles BOA, DOA are congruent $\quad OBA = ODC$, etc

6 Let $ABCD$ be cyclic quadrilateral $\quad$ Make angle DAE = angle BAC, E being on BD
Triangles DAE, CAB are similar $\quad AC\ \ DE = BC\ \ AD$
Triangles BAE, ADC are similar $\quad AC\ \ BE = AB\ \ DC$, etc
In second part, $PABC$ is cyclic $\quad PA\ \ BC = AB\ \ PC + AC\ \ PB$, etc

No. 90

1. $BC > BQ > PQ$.
3. Perpendicular from P on $AB < AP <$ side of square.
Angle $APQ = 120°$; angle $APD = 75°$. $\therefore DPQ = 45°$.
4. Triangles PAQ, HBK are congruent (chords PQ, HK equidistant from centres).
$\therefore$ angle APQ = angle BHK. $\therefore AP$ and BH are parallel. $\therefore PH = AB$, etc.
Or, suppose circles coincide; as B circle is moved and centre travels from A to B, H travels from P to H and K from Q to K. $\therefore PH = AB = QK$.
5. $AE : EB = AD : DB = AD : DC = AF : FC$. $\therefore EF$ is parallel to BC.
6. Let fall AZ perpendicular to XY, along XY measure $ZB = 2{\cdot}5$ cm. Angle ABX is 31°.

No. 91

1. Triangles BAD, CAD are congruent. $\therefore$ angle BAC is bisected by AD produced.
2. By proof of concurrency of the medians, lines respectively equal to $\frac{2}{3}$ of the medians form a triangle; hence any two medians are greater than the third.
Produce AD to E, making $DE = AD$. Then $AC + CE > AE$; i.e. $AC + AB > 2AD$. $\therefore$ sum of medians less than perimeter of triangle.
3. Locus is line parallel to XY, 1·5 cm. from XY, between O and XY.
4. Draw AD perpendicular to BC, then $BD = DC$.
$\therefore AB^2 - AO^2 = BD^2 - OD^2 = BO \,.\, OC$.
5. Line bisecting angle between two lines bisects angle between perpendicular drawn to lines at angular point. Hence PAQ bisects an angle between the radii drawn to A. Let X be centre of P circle and Y of the Q circle. Then triangles PXA, QYA are equiangles. $\therefore PA : AQ = AX \; AY$.
6. (i) Triangle AOB . triangle $AOC = AB : AC$.
$\therefore AO \,.\, OB \sin AOB : AO \,.\, OC \sin AOC = AB : AC$.
(ii) Triangle AOB + triangle BOC = triangle AOC, etc.

No. 92

2. Along BC measure $BD = 2$ in., draw CE, parallel to DA, cutting BA at E. Construct triangle PQR having $PQ = BD = 2$ in., $PR = BE$, $QR = DE$.
3. Draw PE perpendicular to BC, and bisect BC at D.
$\cdot 7 = PB^2 - PC^2 = BE^2 - EC^2 = (BE + EC)(BE - EC) = 2 \times 2{\cdot}1 \times DE$.
$\therefore DE = \frac{1}{6}$ of 1 in. E is between D and C and perpendicular at E to BC is the locus of P.
4. Suppose circles touch externally. On side of given line opposite to given circle draw a line XY parallel to given line at distance = radius of given circle. Then centre of any circle touching externally is equidistant from XY and the centre of given circle. Similarly if circles touch internally, etc.
5. Join B mid-point of line joining centres to A, required line is perpendicular to AB.
On AC and BC as diameters draw circles. Through C draw PCQ, cutting circles at P and Q and bisected at P. Then PCQ is middle line of square and square can be completed.
6. Triangles PAB, PAC are between same parallels. $\therefore$ areas are as $PB : PC$. Triangles are equiangles. $\therefore$ areas are as $AB^2 \,.\, AC^2$.

No 93

1 $AC = 926$ in , $BO = 1\ 426$ in

2 $EA = FG = HB$. $EAHB$ is a parallelogram

Hence $FGHB = EAHB = 2$ triangle $AEB = ABCD$

3 Draw a line OB and in it take any points A and C, C being between A and B Let $OA = a$, $OB = b$, $OC = c$ We have to prove that

$$OA\ \ CB + OB\ \ AC = OC\ \ AB.$$

By first part, $OA\ \ CB = OA\ \ OB - OA\ \ OC$

and $OB \cdot AC = OB \cdot OC - OB\ \ OA$

and $OC\ \ AB = OC\ \ OB - OC\ \ OA$

4 DE is bisected at right angles by AC and DF is bisected at right angles by AB See Paper 30, Question 6

5 Let ratio be $l\ \ m$, and A, B the two given points Divide AB internally at P so that $AP\ \ PB = l\ \ m$ and externally at Q so that $AQ\ \ QB = l\ \ m$ Circle on PQ as diameter is required locus

6 Triangles BRP and PQC are equiangular and therefore similar

$SP\ \ SB = PQ\ \ BR = QC\ \ PR = SC\ \ SP$

No. 94

1 If XOY is obtuse, angles YOA, XOB are equal bisectors do bisect angles between perpendiculars If XOY is acute, produce XO to Z, then $YOA = ZOB$, etc

2 Triangles POA, QOC are congruent

Produce AC to D so that $CD = CA$, and BC to E so that $CE = CB$ Then D and E are on the other sides of rhombus

On AB describe a segment on side remote from C, containing an angle 60°, complete circle and bisect minor arc at F and produce CF to cut major arc at P P is one vertex of rhombus Produce PC to R so that $CR = CP$ R is the opposite vertex Produce PA, RE to meet at Q and PB, RD to meet at S $PQRS$ is the rhombus Side of rhombus is 3 3 in

3 Let AB be a given line, at B draw BC perpendicular to AB, equal to AB With centre O, the mid point of AB, radius OC, draw circle cutting AB produced at P

4 If A is centre of PQX and B of PQY angles APX, BPY are equal, since each equals $APY - 90°$, hence angles PAX QBY are equal

Also $PQY = \frac{1}{2}PBY$ and $PQX = \frac{1}{2}(360 - PAX)$ $PQY + PQX = 180°$

5 Triangles PAX, PBY are equiangular

6 Height = 136 3 ft

No. 95

1 Let AG perpendicular to DE, cut BF at H

Triangles ADE, ABF are congruent

angle $AFB = AED = HAF$ $HF = HA$

Also angle $HAB = ABH$ $HB = HA$ FB is bisected at H

2 Angle $PBQ = \frac{1}{2}$ right angle $= PQB$ $PQ = PB$

. $AP^2 + PB^2 = AP^2 + PQ^2 = AQ^2$

$AP^2 + PB^2$ is least when AQ is least, i e when P is at O the mid point of AB.

3 Also $AP^2 + PB^2 = AQ^2 = AC^2 + CQ^2 = 2AO^2 + 2OP^2$

4 See Euclid III, Prop 21

5. See Paper 52, Question 5.

Produce AD the perpendicular from vertex of an isosceles triangle ABC to meet circumcircle at E. Triangles ABD and ABE are similar and

$$AE \, . \, AD = AB^2.$$

6. Distance from vertex $= \frac{1}{2} \tan 72 = 1 \cdot 5389$.

Angle of pentagon $= 108°$, and of hexagon.

Let $ABCDE$ be pentagon and $CDFGHK$ be hexagon.

Then CB and DE fall inside hexagon.

Distance from CD of A is $1 \cdot 5389 \times CD$. Distance of HC from CD is $1 \cdot 732 \times CD$. $\therefore$ A falls inside hexagon.

No. 96

1. Proof is same as that for proving that exterior angle of a triangle is greater than an interior opposite angle.

2. Draw OZ equal and parallel to AB. Draw ZC parallel to OX, meeting OY at C. Draw CB parallel to ZO, meeting OX at D. $ABCD$ is required parallelogram.

3. Equate two expressions for $OA^2 + OB^2 + OC^2$.

$AZ^2 + BX^2 + CY^2 = AO^2 + BO^2 + CO^2 - OZ^2 - OX^2 - OY^2$.

Let P, Q, R be mid-points of BC, CA, AB respectively.

Then $AO^2 + BO^2 = 2AR^2 + 2RO^2$.

Hence $AZ^2 + ZB^2 + BX^2 + XC^2 + CY^2 + YA^2$

$$= 2(AO^2 + BO^2 + CO^2) - 2(OX^2 + OY^2 + OZ^2)$$
$$= 2(AR^2 + BP^2 + CQ^2) + 2(RO^2 - OZ^2 + OP^2 - OX^2 + OQ^2 - OY^2$$
$$= \text{constant} + 2(RZ^2 + PX^2 + QY^2).$$

$\therefore$ L.H.S. is a minimum when O is centre of circumcircle.

5. $BP : PC =$ triangle AOB : triangle AOC, etc.

6. (i) Triangles PRQ, ADC are similar.

$$\therefore \frac{AB}{RQ} = \frac{AD}{PQ} = \frac{AC}{PR}.$$

Also $RQ \, . \, AP + PQ \, . \, AR = PR \, . \, AQ$. (Ptolemy's Theorem, Paper 89, Question 6.)

$\therefore AB \, . \, AP + AD \, . \, AR = AC \, . \, AQ$.

(ii) $AD^2 + AB^2 = AC^2$. $\therefore AD \, . \, AR + AD \, . \, DR + AB \, . \, AP + AB \, . \, BP$
$$= AC \, . \, AQ + AC \, . \, CQ.$$

$\therefore AB \, . \, BP + AD \, . \, DR = AC \, . \, CQ$.

No. 97

2. 2·8 in.

3. See Paper 20, Question 2.

Let XYZ be any points in BC, CA, AB respectively.

If XZ, YZ are not equally inclined to YZ, find O so that OX, OY are equally inclined to AB. Then $OX + XY + YO < ZX + XY + YZ$. Hence if the sides of XYZ are not equally inclined at X, Y, Z to the sides BC, CA, AB respectively, a triangle of smaller perimeter can always be found. Hence the triangle formed by joining the feet of perpendicular has the minimum perimeter.

4. Bisector of angle and perpendicular bisector of side both bisect same arc. AD is common chord of the two circles.

5. See Paper 39, Question 4.

6. (i) Square = rectangle 5 in. by 3 in.; call side x. (ii) Call side of square y. Draw new equilateral triangle by increasing side 1 in. in ratio $x : y$. Side of required triangle is = 5·9 in.

No. 98

1 *Draw* isosceles triangle ABC having $AB = AC = 3$ in., $BC = 2\ 2$ in Drew AH perpendicular to BC and produce AH to K, making $AK = 5$ in, etc

2 If P is between B and D, the mid point of BC, bisect AC at E Drew BQ parallel to PE to meet AC at Q, then Q is between E and A and PQ is bisector of triangle ABC

3 Diagonals AC, BD bisect at right angles at O

$\therefore PA \cdot PC = PO^2 - AO^2 = PB^2 - AB^2$

4 In inscribed triangle side $= 6 \sin 66° = 5\ 48$ cm

In describing triangle side $= 3\ (\tan 73\frac{1}{2}° + \tan 49\frac{1}{2}°) = 13\ 64$ cm

5 *See Paper 89, Question 6*

$ACDE$ is cyclic $\therefore AD\ \ CE = AE\ \ CD + DE\ \ AC$, etc

Or by congruent triangles

6 $6\ 8^2 - 4\ 2^2 = 10 \times 1\ 6 = 4^2$ $\therefore$ triangle is right angled

Angles are 90°, 46° 24′, 43° 36

No. 99

1 Triangles AOB, POQ are congruent $\therefore$ angles OBA, OQP are equal $OB = OQ$ $\therefore$ angles OBQ, OQB are equal Hence angle VBQ = angle VQB

2 With centres C and D, radii each $2\ 5$ in, strike arcs cutting DA and DA produced at Q and P respectively Then parallelogram $PBCQ$ = parallelogram $ABCD$ At B and P make angles = 110°, etc Other sides $= 2\ 47$ in

3 Angle $QEP = 2\,C$, E is centre of circle APC

F is centre of circle $AQPB$ $\therefore$ angle $QFP = 2^{ce}$ angle $QAP = 180 - 2C$

$\therefore$ angles QEP, QFP together equal 180°

4 At any point X on A circle draw tangent $XH = 3$ cm

At any point Y on B circle, draw tangent $YK = 4$ cm

Circles centre A, radius AH, and centre B, radius BK, intersect at points P.

5 Angle $AQB = 180 - ACB$ (cyclic quadrilateral) Angle $PQB = \frac{1}{2}ACB$

$\therefore$ Angle $PBQ = 180 - (180 - ACB) - \frac{1}{2}\,ACB$ $\therefore PB = PQ$

6 Let X, Y, Z be centres of escribed circles touching BC, OA, AB respectively, and let I be centre of inscribed circle

$ZBCY$ is cyclic quadrilateral $\therefore$ angle XZI = angle XYI But XI is common chord of circles XZI, XYI and they subtend equal angles at the circumference Hence circles XZI, XYI are equal, etc

No. 100

1 Make angle $BCD = 45° - \dfrac{B}{2}$

2 Triangles BOA, BCD are congruent $\therefore$ triangles BEA, BED are congruent.

3 $ALKB$ is a fixed circle with diameter AB Angle LAK is fixed in magnitude $\therefore LK$ is of constant length

4 Bisect AB at E, draw EP parallel to BC

5 *Reductio ad absurdum*

Angle $DBC = ACB = BAC$ $\therefore DB$ is a tangent.

$DE\ \ DC = DB^2$ $\therefore AB\ \ DE = AC^2$

6 Triangles OAC, ODB are congruent, since A, B D, O are concyclic.

$\therefore$ angle EAB = angle OBD and triangles EAB OBD are similar

$\therefore$ angles BOF, FBO are equal and $OF = BF$

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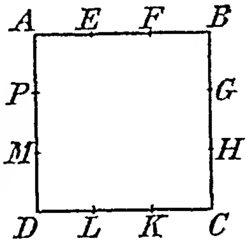
No. 13

1. In the figure the sides of a square $ABCD$ are each divided into three equal parts, E, F, G, H, K, L, M, P being the points of trisection. Prove that E, G, K, M are the corners of a square.



2. In the same figure determine what fraction the area of the octagon $EFGH$ $KLMP$ is of the area of (i) the square $ABCD$, (ii) the square $EGKM$.

3. State and prove the geometrical proposition corresponding to the algebraical formula $a(a+b) = a^2 + ab$.

Four points A, B, C, D are taken in that order on a straight line, show that—

$$AB . CD + AD . BC = AC . BD.$$

4. Show how to draw two tangents to a circle from a point outside it. Prove that the parts of these tangents between the point and the circle are equal.

If the sides AB, BC, CD, DA of a quadrilateral each touch the same circle, prove that $AB + CD = BC + DA$.

5. On a line 2·3 in. long describe a segment of a circle to contain an angle of 70°. Measure the radius.

What is the area of the largest triangle that has base 2·3 in. long and the opposite angle 70° ?

6. C is a fixed point on a diameter AB of a given circle; PCQ is any chord through C; at C a perpendicular is drawn to AB which meets AP, AQ (produced if necessary) in R and S. Prove that P, Q, R, S lie on a circle and that the rectangle $CR . CS$ is of constant area.

No 14

1 State a construction, with proof, for drawing a line through a given point parallel to a given line

Draw a line 5 5 in long, and by means of parallels divide it into six equal parts

2 In a triangle ABC the side AB is greater than the side AC, the bisector of the angle A meets BC at D Prove that the angle ADB is an obtuse angle

3 State and prove a rule for finding the area of a trapezium Construct a trapezium $ABCD$, having AD parallel to BC, $AD = 4\ 8$ cm, $BC = 3\ 2$ cm, angle $A = 42°$ and angle $D = 68°$ Find its area

4 Define a circle, and prove that—

(i) A straight line cannot meet a circle in more than two points

(ii) Two circles cannot cut in more than two points

5 If two chords of a circle intersect within the circle, prove that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other

By a geometrical construction find the value of the fraction $\dfrac{3\ 4 \times 2\ 3}{2\ 8}$, correct to one decimal place Verify by calculation

6 Any point P is taken on the circumference of a circle of which AB is any diameter PB meets the radius OC, perpendicular to AB, at R, and the tangent at P meets the radius OC produced at Q Prove that $QP = QR$

No. 15

1. Prove that the sum of the diagonals of a quadrilateral is less than the sum of the sides of the quadrilateral, but greater than half the sum of the sides.

2. Draw two triangles to show that two triangles may have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, and yet not be equal in all respects.

The diagonal AC of a quadrilateral $ABCD$ bisects each of the angles BAD, BCD. Prove that it bisects the diagonal BD.

3. Through a point O on the diagonal AC of a parallelogram $ABCD$, HOF, EOG are drawn parallel to the sides, H being in AB and E in BC. Prove that the parallelograms HE, FG are equal in area. Prove also that HG and EF are parallel.

4. If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angles supplementary, prove that the triangles are equal in area.

Squares $ABDE$, $ACFG$, $BCHK$ are described on the sides of a triangle ABC with a right angle at C. Prove that the triangles AGE, BKD are equal in area.

5. Of all lines drawn from a point within a circle to the circumference, which are the greatest and least? Of all chords that can be drawn through a point within a circle which are the greatest and least?

On OA a radius of a circle with centre O and radius 5 cm., a point P is taken 2 cm. from O. With P as centre a circle is described with radius 1 cm.; draw the longest and shortest chords of the big circle that will touch the small circle. Prove the accuracy of your construction.

6. A straight line PQ of constant length slides so that P moves along a line OX, and Q along a line OY. Find the locus of the centre of the circumcircle of the triangle POQ.

No. 16

1 Two isosceles triangles are described on the same base and on the same side of it. Prove that the line joining the vertices, when produced, bisects the base at right angles.

2 A bar AB, 4 ft. long is suspended in a horizontal position by ropes AC, BD, respectively 8 ft. and 6 ft. long, from two hooks in the ceiling CD 10 ft. apart. By drawing a figure to scale find the depth of AB below the ceiling.

3 Draw a triangle OAB with sides $OA = 2$ in., $OB = 2·4$ in., angle $AOB = 35°$, then draw a triangle OPQ with sides $OP = 2$ in., $OQ = 2·4$ in., angle $OPQ = 35°$ and having OP inclined at 45° to OA. Verify, and prove by argument, that the angle between PQ and AB is also 40°.

4 Give a construction for drawing the tangents to a circle from an external point and justify your construction.

If the two tangents drawn from P to a circle with centre O touch the circle at A and B prove that PO bisects AB at right angles.

5 State how to find the centre of the circumcircle of a triangle.

Perpendiculars BE, CF are let fall on the opposite sides of a triangle ABC; prove that the triangles BFC, EFC have the same circumcentre.

6 From a point P two tangents PA, PB are drawn to a circle, and the perpendicular from P to AB meets the circle at C and D. Prove that AC bisects the angle PAB and that AD bisects the angle between AB and PA produced.

No. 17

1. The side BC of a triangle is bisected at D and AD is produced to E so that $DE = AD$. Prove that $CE = AB$.

If AC is produced to any point X, prove that the angle BCX is greater than the angle ABC.

2. Draw a triangle ABC having angle $A = 48°$, angle $C = 63°$, $AB = 7$ cm. In it find a point P such that $PA = PB$, and angle PAB = angle PBC.

3. State and prove Pythagoras's theorem.

In a quadrilateral $ABCD$ the angles at A and B are each a right angle, and $AB = AD + BC$. Prove that $CD^2 = 2(AD^2 + BC^2)$.

4. Prove, completely, that the angle at the centre of a circle is double any angle at the circumference standing on the same arc.

Any point P is taken on the circumference of a circle with centre O, and is joined to the extremity A of a diameter AB. If PA is produced to Q, prove that the angle BAQ is half the reflex angle BOP.

5. Prove that the perpendicular from the centre of a circle to any chord bisects that chord.

A and B are any two points on a circle, show how to draw two parallel chords AP, BQ such that $AP = 2BQ$.

6. An equilateral triangle ABE is described on the side AB of a square $ABCD$ and on the same side of it as the square. The line from A at right angles to DE meets CD at F. Prove that a circle can be described about $AEFD$.

Show also that $DF = CD(2 - \sqrt{3})$.

No 18

1 On the sides AB AC of a triangle equilateral triangles ABE ACD are described outside the triangle ABC Prove that $BD = CE$

2 If three parallel straight lines cut off equal intercepts on one transversal prove that the intercepts on any other transversal are also equal

Two straight lines AX AY contain an angle 55° a point P is taken within the angle XAY 1 5 cm from AX and 2 cm from AC Construct a straight line through P so that the part intercepted between AX and AY is bisected at P

State and prove your construction

3 Prove that a diagonal of a parallelogram always bisects the area but in general does not bisect the angles of the parallelogram

In a trapezium $ABCD$ the sides AB and DC are parallel and AB is less than DC Through A a line is drawn parallel to BC meeting DC at E and DE is bisected at F Prove that the triangle BCF is half the trapezium

4 The *opposite sides* AB and DC *of* an *irregular quadrilateral* meet when produced at E The circumcircles of the triangles BEC ADE meet again at F Prove that the angles BFC AFD are equal

5 At the ends A and B of a diameter of a circle tangents are drawn these tangents are cut by a third tangent at C and D respectively Prove that the rectangle AC BD is equal to the square on the radius of the circle

6 Divide a straight line AB 6 cm long into two parts at a point P so that the rectangle contained by the whole line AB and the part BP shall equal the square on the part AP State your construction and prove its validity Also draw any circle to pass through A and P from B draw a tangent BQ to this circle With centre B and radius BQ draw a circle cutting BA at R Prove that $AB = BP$

No. 19

1. The diagonal DB of a parallelogram $ABCD$ is produced to E, so that $BE = DB$, and the parallelogram $CBEF$ is completed. Prove that AB and BF are in the same straight line.

2. In a triangle ABC with a right angle at C, any point D is taken in BC, and any point E in AC; prove that DE must be less than AB.

Two right-angled triangles ABC, DEF have their hypotenuses AB and DE equal, but AC is less than DF. Prove that angle ABC is less than the angle DEF.

3. Construct a quadrilateral $PQRS$, having $PQ = 2{\cdot}4$ cm., the diagonal $QS = 3{\cdot}2$ cm., angle $QPS = 66°$, angle $SQR = 39°$, and angle $QSR = 68°$.

Make a triangle equal to it, and test the accuracy of your construction by calculating both areas.

4. What is meant by the *projection* of a finite straight line on another?

Show that in any triangle the difference of the squares on any two sides of a triangle is equal to twice the rectangle contained by the third side and the projection on it of the median bisecting that side.

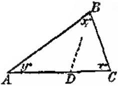
5. Write out two enunciations in which the word *alternate* is used, and explain the meanings of the word.

The railway line between two places X and Y consists of two circular arcs, XZ and ZY, which have a common tangent PZQ, the arcs being continuous and on opposite sides of the tangent. Draw a plan of the line, scale 1 cm. to the mile, being given that the chord ZZ is 3 miles long, the angle XZP is 30°, the angle YZQ is 25° and the radius of the arc YZ is 5 miles. What is the shortest distance from X to Y?

6. P and Q are two points 4·5 in. apart; draw a straight line such that the shortest distance from P to it shall be 3 in., and from Q 2 in.

No 20

1 If an isosceles triangle ABC has angles x° and y° as in the figure, determine the value of x and y so that it may be possible to draw a line BD to divide the triangle into two isosceles triangles ADB, CDB Prove that AD is greater than DC



2 Two points A and B are on the same side of a straight line XY, at unequal distances from it Show with proof how to find a point C in XY such that the angle ACX = the angle BCY

If P be any other point in XY, prove that—

$$AC + CB < AP + PB$$

3 Prove that the area of a triangle is one half the area of the rectangle on the same base and with the same altitude

On the side AC of a triangle ABC right angled at C, an equilateral triangle ADC is described Prove that the triangle BCD is half the triangle ABC

4 Show that four circles can be drawn to touch three straight lines each of which cuts the other two

The centre of the inscribed circle of a triangle ABC is I, and the centre of the circle touching BC and the sides AB and AC produced is E Prove that the circle described on IE as diameter passes through B and C

5 A point D is taken in a side BC of an equilateral triangle ABC, and an equilateral triangle CDE is described on CD, the vertices A and E being on opposite sides of BC, and AD is produced to meet BE at F Prove that the circumcircle of the triangle BDF touches AB

6 Prove that the perpendiculars from the vertices of a triangle upon the opposite sides meet in a point

No. 21

1. Prove that two right-angled triangles are congruent if they have (i) the hypotenuse and an acute angle of the one equal to the hypotenuse and an acute angle of the other, or (ii) the hypotenuse and another side of the one equal to the hypotenuse and another side of the other, each to each.

2. Draw the locus of all acute-angled triangles on a base 3·5 cm., having area 7 sq. cm.

In the same figure draw the particular triangle that has the smallest perimeter and satisfies the conditions. Give your reasons.

3. Explain how to construct a square equal to a given rectangle.

Divide a straight line of length 4·3 cm. internally, so that the rectangle contained by the segments is of area 3 sq. cm. State your construction, which must be entirely geometrical, and not depend on an arithmetical extraction of a square root.

4. State, with proof, how to find the centre of a given circular arc.

Find, to the nearest millimetre, the radius of the arc in the adjoining figure.

5. If the opposite angles of a quadrilateral are supplementary, prove that a circle can be described to pass through its angular points.

The vertex A of an equilateral triangle ABC is joined to a point D on BC produced, and on AD an equilateral triangle ADE is described. Prove that either A, B, D, E, or A, C, D, E, are concyclic.

6. Two circles intersect, one of the points of intersection being P. Explain, with proof, how to draw through P a line QPR, meeting one circle at R and the other at Q, such that QR is bisected at P.

No 22

1 From two points A and B on a straight shore, 150 yds apart, a rock C is seen and the angles CAB, CBA are 45° and 60° respectively Find, by drawing to scale 1 cm = 20 yds, and without using a protractor, the distance of C from the shore

2 Prove, by the theory of parallels that the sum of the angles of a triangle is equal to two right angles

State the similar fact about a rectilinear figure of n sides

Draw any irregular pentagon, having each of its angles obtuse, produce the sides of the pentagon to meet, thus forming a five pointed star Find the sum of the angles at the points of the star

3 The side BC of a parallelogram $ABCD$ is bisected at E, and the side CD at F Prove that the area of the triangle AEF is three times the area of the triangle CEF, and that together they are equal to half the parallelogram

√4 In any circle prove that equal chords are equi distant from the centre

A point P is taken 2 3 in from the centre O of a circle with radius 1 5 in Draw through P a line meeting the circle at A and B so that the length of AB shall be 1 9 in Explain your construction

5 Explain how to draw a circle to pass through two given points and to touch a given straight line

6 ABC is an isosceles triangle in which the angles at B and C are each twice the angle at A, at C an angle BCD is made equal to the angle at A, CD meeting AB at D Calculate the size in degrees of all the angles in the figure, and prove that rectangle BA, BD = square on AD

No. 23

1. State with proof, the construction for bisecting with ruler and compass a given angle.

Without using a protractor draw an angle of 75° and divide it into five equal parts.

2. If a quadrilateral has both pairs of opposite sides equal to one another, prove that its diagonals bisect one another.

3. State and prove the proposition concerning the square on the side of a triangle opposite an obtuse angle.

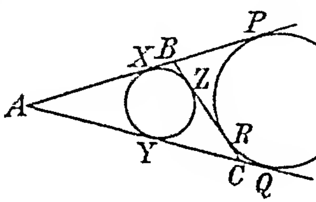
A triangle has sides 1·7 in., 1·3 in., 1·1 in., determine, firstly by drawing, secondly by calculating, whether the centre of the circumcircle is inside or outside the circle.

4. Prove that equal arcs in a circle subtend equal angles at the centre, and that equal chords subtend equal angles at the centre.

If an arc AB is double an arc PQ in the same circle, with centre O, prove that angle AOB is double the angle POQ; but if in a circle a chord AB is double a chord PQ, the angle AOB is more than double the angle POQ.

5. In a quadrilateral $ABCD$ the diagonals AC, BD intersect at E. Construct the quadrilateral being given $AE = 2$ cm., $BE = 1$ cm., $CE = 1·5$ cm., angle $ABD = 100°$, and angle $ACD = 100°$.

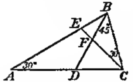
6. In the figure the two circles touch the two lines AB, AC at X and Y, P and Q; and BC is an internal common tangent to the two circles touching at Z and R. Prove that—



(i) $2AP = AB + BC + CA$.
(ii) $2AX = AB + AC - BC$.
(iii) $ZR = AC - AB$.

No. 24

1 In the figure $AB = AC$ and the angles CAB, CBF, BCF have the values shown Prove that $DA = DB$ and $BE = BF$.



2 Define a parallelogram, and prove that its opposite sides are equal

A pair of parallel lines AB and DC intersect another pair of parallel lines AD and BC at the points A, B, C, D A point P is taken anywhere in the plane of these lines Show, with proof, how to draw through P a line such that the part intercepted between one pair of parallel lines is equal to the part intercepted between the other pair

3 The area of a quadrilateral is 24 sq in , and its two diagonals are 6 in and 8 in in length Prove that the diagonals are at right angles, and that the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair

4 Prove that if two circles touch, whether internally or externally, the point of contact lies on the line joining the centres

Draw two circles with radii 4 cm and 5 cm , and with their centres 7 cm apart Draw a third circle with radius 3 cm to touch each of the other two

5 Prove that the angles between a tangent to a circle and a chord through the point of contact are equal to the angles in the alternate segments

Tangents are drawn at two points A and B on a circle, and from a third point P on the circle perpendiculars PH, PK are let fall on the tangents and PL on the chord AB Prove that the triangles PHL, PKL have their angles equal, each to each

6 Construct, without any long calculation of square roots, a triangle whose sides are $\sqrt{5}$ cm , $\sqrt{7}$ cm , $\sqrt{12}$ cm , and measure the largest angle

No. 25

1. Construct a triangle with sides $AB = 4$ cm., $BC = 3$ cm., $AC = 5$ cm. Draw a straight line PQ 6 cm. long; construct, without using a protractor, an angle PQR equal to the angle ACB and an angle QPR equal to angle BAC. Measure PR and QR.

2. The middle points D, E, F of the sides BC, CA, AB of a triangle are joined. Prove that AD is bisected by FE and that the area of the triangle DEF is one-quarter of the area of the triangle ABC.

3. Give the enunciation which states by how much the square on the side subtending an acute angle exceeds the sum of the squares on the sides containing that angle.

In a triangle ABC, AD is the median bisecting BC, and AX is at right angles to BC; prove the difference between the squares on AB and AC is equal to twice the rectangle $BC \, . \, DX$.

Write out the general enunciation corresponding to this particular enunciation.

4. Prove, fully, that the middle point of the hypotenuse of a right-angled triangle is equidistant from the three angular points.

AB is a diameter of a circle of radius 3·7 cm., and AC is a chord 2 cm. long. Without any calculation, construct a rectangle equal in area to the square on AC and having one side equal to AB. Give a proof.

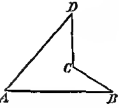
5. In a circle, centre O, radius 1·5 in., inscribe a quadrilateral $ABCD$ having $BC = CD$, the angle $ABC = 95°$ and the angle $AOB = 40°$. State the steps of your construction.

Calculate the number of degrees in the angle BCD, and so check the accuracy of your figure.

6. Find the locus of a point which moves so that the tangents drawn from it to two fixed circles are equal. [Consider separately two cases, (i) the circles intersect, (ii) the circles do not intersect.]

No 26

1 Copy the *quadrilateral ABCD by measuring* any two angles and as few lengths as possible Give the enunciations of the congruency propositions that justify your construction



2 A triangle ABC has the sides AB and AC equal, the angles at B and C are bisected by lines meeting the opposite sides at P and Q Prove that PQ is parallel to BC

3 Prove that a parallelogram and rectangle on the same base and between the same parallels are equal in area

Squares $AGFC$, $BCDE$ are described on the sides AC, BC of a *triangle right angled at A* DC *is produced to meet* FG produced at H, and a line through A parallel to DH meets ED at K and FG produced at L From this figure prove that the square AF = rectangle CK

4 Prove that any two angles at the circumference of a circle standing on the same chord are either equal or supplementary

$ABCD$ is a cyclic quadrilateral and X is the middle point of the arc BD on the side of CD, remote from A Prove that XC bisects the angle between DC and BC produced to E

5 Two circles with centres P and Q touch externally at A, an external common tangent touches the circles at B and C respectively, and meets the common tangent drawn through A at D *Prove that the angles BAC and PDQ are* each equal to a right angle

6 Show, with proof, how to produce a chord AB of a circle to a point P, such that rectangle $AP \cdot PB$ = square on AB

No. 27

1. Two triangles ABC, ADE have $AB = AD$, $AC = AE$, $BC = DE$, and are placed so that the line CE cuts both AB and AD without being produced. Prove that CE is parallel to BD.

2. The angles A, B, C of a quadrilateral $ABCD$ are respectively 100°, 70° and 150°. A new quadrilateral is formed by bisecting the exterior angles of $ABCD$. Find the sizes of the interior angles of this new quadrilateral.

3. If the square on one side of a triangle is equal to the sum of the squares on the other two sides, prove that the triangle is right-angled.

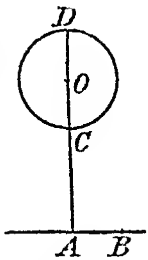
Find the area of a triangle whose sides are 17·8 in., 16 in., 7·8 in.

4. Write the enunciation connecting the squares on the sides of a triangle with the square on one of the medians.

In a triangle ABC, the angle C is a right angle, and AD and BE are medians. Prove that $4AD^2 + 4BE^2 = 5AB^2$.

5. Reproduce the adjoining figure taking $OA = 3$ in., $AB = 1$ in., and radius of circle to be 1 in.

Construct two circles each of which touches the given circle, and also touches the line AB at B. Prove your construction and measure the distance between the centres.



6. The diagonals of a trapezium $ABCD$, in which AB is parallel to CD, intersect at E; prove that the circles described about the triangles ABE, CDE touch. If the circles described about the triangles ADE, BCE. also touch, prove that $ABCD$ must be a parallelogram.

No. 28

1 A line AB of definite length is moved to any other position $A'B'$ in the same plane, the perpendicular bisectors of AA' and BB' meet at O Prove that the angle AOB is equal to the angle $A'OB'$

2 Show that parallelograms on equal bases and between the same parallels are equal in area

$ABCD$ is a parallelogram, equal lengths AP, DQ are cut off from the sides AB, DC Any two parallel lines are drawn through P and Q, show that the parallelogram they intercept between AD and BC is always of the same area as $ABCD$

3 In a triangle ABC, a perpendicular AD is drawn from A to BC If $AB = 65$ cm, $AD = 60$ cm, $AC = 156$ cm Show that the triangle is right-angled

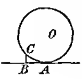
4 If two circles intersect prove that the line joining their centres, produced if necessary, bisects the common chords at right angles

Being given a circle of radius 4 cm, draw two other circles of radius 2 cm so that they intersect the first circle in the same two points the common chord being of length 3 cm

5 Prove that a line passing through the centre of a circle perpendicular to a chord bisects the chord

The longest line that can be drawn from a point P inside a circle to the circumference is 8 cm long the shortest is 2 cm Construct the figure Measure the length of the chord that is bisected at P, and verify by calculating the length of that chord

6 The figure represents a wheel of radius 2 ft 1 in, which is just about to roll, without slipping, over a thin obstacle BC of height 10 in Calculate the length of BA



In a figure drawn to the scale of 1 cm = 10 in, trace out the loci described by O and A respectively, while the wheel is surmounting the obstacle

No. 29

1. Prove—

(i) The vertex of an isosceles triangle lies on the perpendicular bisector of the base.

(ii) All points equidistant from two fixed points lie on the perpendicular bisector of the line joining the points.

(iii) The centres of all circles passing through two given points lie on the perpendicular bisector of the line joining the points.

2. A line PQ is bisected at R; from P, Q, and R perpendiculars PH, QK, RL are let fall on another straight line passing through a point O, not between H and K. Prove that (i) $2OL = OH + OK$, (ii) $2RL = PH + QK$.

If a line AB is divided equally at O and unequally at P, prove (i) geometrically, (ii) algebraically, that OP is equal to the difference between AP and PB.

3. Construct a quadrilateral $ABCD$ having $AB = 2{\cdot}3$ cm., the diagonal $BD = 3$ cm., angle $BAD = 65°$, angle $DBC = 40°$, and the angle $BDC = 70°$. Make a triangle equal to it, and prove your construction to be correct.

Making the necessary measurements, calculate the areas of the quadrilateral and triangle.

4. AOB is a straight line bisected at O; a point P moves so that $AP^2 + BP^2$ is constant. Prove that the locus of P is a circle.

Draw the circle when $AB = 3$ in. and the constant is $8\frac{1}{2}$ sq. in.

5. Any point D is taken in the side AB of a triangle ABC; through D a line is drawn, meeting AC at E and making the angle ADE equal to the angle ACB. Prove that the rectangle $AC \,.\, AE$ is equal to the rectangle $AD \,.\, AB$.

6. Without using a protractor, construct on a base BC 1 in. long an isosceles triangle ABC having the angle A equal to 36°.

No 30

1 If two triangles have two sides of one equal, each to each, to two sides of the other, but the angle contained by the two sides of the first triangle greater than the angle contained by the corresponding sides of the other, prove that the third side of the first is greater than the third side of the other

Two triangles ABC, AED have AD, AC in the same straight line, and AD less than AC, $AB = AE$, and the angles at D and C are right angles but on opposite sides of AC Prove that the angle DAE is greater than the angle CAB

2 What do you know about the line joining the middle points of two sides of a triangle?

The base BC of an isosceles triangle ABC is bisected at D and AC, one of the equal sides, at E, the lines BE, DE are bisected at H and K respectively Prove that $DH = DK$ and $AH = AK$

3 Construct a rhombus of side 2 5 in equal in area to a square of side 2 in

Check your figure by measuring the diagonals of the rhombus and calculating the area

4 On a base 2 5 in long construct a segment of a circle to contain an angle of 100°

A cyclic quadrilateral $ABCD$, the diagonals of which intersect at O, is such that $AOB = 140°$ $AB = 2 5$ in $BC = 1$ in, $AC = 2 5$ in Construct the quadrilateral and measure OD.

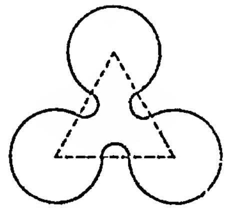
5 A point P is taken 4 cm from the centre of a given circle with radius 2 5 Draw through P a straight line which will cut from the given circle a segment containing an angle of 100°

6 The perpendicular let fall from the vertex A of a triangle ABC to the side BC meets BC at D, and the circumcircle at E the perpendicular from B to AC meets AD at K Prove that $KD = DE$

No. 31

1. Two straight lines AB, CD bisect each other at O; AC, BD are bisected at E, F respectively. Prove that EF passes through O.

2. Draw an equilateral triangle with side 4 cm., and then draw accurately a figure similar to the adjoining figure.



3. By a geometrical construction obtain an approximate solution of the equations $x + y = 10$, $xy = 20$. Explain and justify your method.

4. (i) A, B, C, D are four points on the circumference of a circle. If AC and BD bisect one another, prove that they both pass through the centre.

(ii) A diameter AB of a circle bisects a chord CD. If BC is parallel to AD, prove that CD bisects AB.

5. If a line subtends equal angles at two points on the same side of it, prove that the two points and the extremities of the line lie on a circle.

AB, BC, CD are three straight lines, of which AB and CD are equal and on the same side of BC, and the angle ABC is equal to the angle BCD. Prove that the four points A, B, C, D lie on a circle. Prove also that AD is parallel to BC.

6. A circle is described about an equilateral triangle ABC, and another equilateral triangle ABD is described on the other side of AB; with centre D and radius DA a circle is described. A point P is taken on the circumference of the first circle; PA and PB, produced if necessary, meet the second circle at Q and R respectively. Prove (i) that QR is a diameter of the second circle, (ii) that S the intersection of AR and BQ lies on the first circle.

No 32

1 $ABCD$ is a rectangular croquet lawn the width AD being 40 ft On the lawn are three balls P Q R P is 10 ft from AB and 5 ft from AD Q is 20 ft from A and 30 ft from D R is equidistant from P and Q and also equidistant from AD and AB

Draw a figure to a convenient scale and so find the distances of P from AB and AD

2 Define *parallel* straight lines Give without proof a practical test for finding whether two given lines may be considered parallel

If all the angles of a hexagon are equal prove that three pairs of sides are parallel

3 What is meant by the *orthocentre* of a triangle? Through the angular points of a triangle ABC lines are drawn parallel to the opposite sides these parallels forming a triangle PQR Prove that the orthocentre of ABC coincides with the circum centre of PQR

4 AB is a chord of a circle centre O the perpendicular bisector of OB meets AB at C and CO is produced to any point D Prove that the angle AOD is three times the angle BOC

5 State and prove a construction for drawing a circle to touch the side BC of a triangle ABC and the sides AB AC produced

If this circle touches AB and AC produced at P and Q respectively prove that $AP = s$ when s is the semiperimeter of the triangle ABC

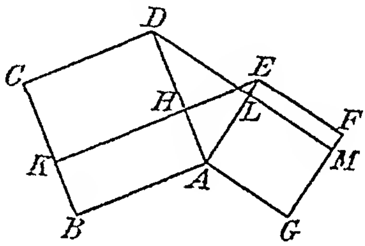
6 An endless belt passes round two wheels one 5 ft and the other 2 ft in radius their centres being 9 ft apart By drawing an accurate figure to the scale 1 cm = 1 ft find the length of the portions of the belt (assumed to be straight) between the wheels Verify this by calculation

No. 33

1. BAC is any angle and $AB = AC$. From AB, AC respectively equal lengths AP, AQ are cut; CP and BQ intersect in X. Prove that AX bisects the angle BAC.

2. Draw a trapezium $ABCD$, being given that AB and DC are parallel and $AB = 2{\cdot}4$ in., $BC = 3{\cdot}6$ in., $CD = 4{\cdot}5$ in., $DA = 3$ in. State the steps of your construction.

3. Write out in full the enunciations which connect the square on one of the sides of a triangle with the squares on the other sides.



Two squares $ABCD$, $AGFE$ are placed as in the figure; EHK is parallel to AB and DLM is parallel to AG. Prove that the rectangles $ABKH$ and $AGML$ are equal, (i) by proving them to be double equal triangles, (ii) by using one of the propositions mentioned in the first part of this question.

4. Show that a square is the only parallelogram which can be inscribed in a circle and also have a circle inscribed in it.

5. If from a point P outside a circle a tangent PA and a secant PBC are drawn, show, without using the properties of similar figures, that $PA^2 = PB \cdot PC$.

A point P is taken on the produced common chord of two circles; PT is drawn to touch one of the circles and PQR is a secant of the other. Prove that the circumcircle of the triangle QRT touches one of the original circles.

✓6. Draw a circle of $2{\cdot}3$ in. radius and take a point A on the circumference. Inscribe a triangle ABC in the circle having $A = 60°$, and $B = 75°$.

Give reasons for your construction and measure the sides of the triangle.

No. 34

1 In a triangle ABC the side AB is greater than AC, the bisector of the angle A meets BC at D Prove that BD is greater than DC

2 Prove that triangles of equal area, on the same base and on the same side of it, lie between the same parallels

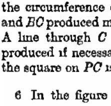
Draw the complete locus of the vertices of all acute angled triangles having a given BC 6 cm long for base, and area 6 sq cm Give your reasons

3 The diagonal AC of a parallelogram $ABCD$ is produced to E so that $AE = 2AC$, and DC is produced to meet EB at F Prove that F is the mid point of BE and that $DC = 2CF$

4 Construct a triangle ABC being given that the radius of the circumcircle is 1 5 in, $BC = 2$ in, and $CA = 1$ 2 in

5 Two circles intersect at A and B, a point C is taken on the circumference of one of the circles inside the other, AC and BC produced meet the second circle at D and E respectively A line through C parallel to DE meets AE at P, and EA, produced if necessary, meets the first circle at F Prove that the square on PC is equal to the rectangle PA, PF

6 In the figure $PQRSTVWX$ represents a regular octagon inscribed in the square $ABCD$, prove that $XR = PS$



Hence, or otherwise, find by a geometrical construction the side of a regular octagon inscribed in a square of side 4 cm

If x in is the side of a regular octagon inscribed in a square of side a in, prove that $x(1 + \sqrt{2}) = a$

No. 35

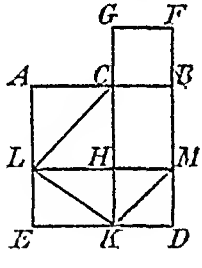
1. From the ends of a finite straight line AB equal parts AC and BD are cut. Through A and D parallel lines AE and DF are drawn; through C and B parallel lines CE and BF are drawn, meeting the other parallels in E and F respectively. Prove that AF and DE bisect one another.

2. Construct a triangle ABC having $BA = 2{\cdot}6$ in., $AC = 3{\cdot}2$ in., angle $ABC = 55°$. Measure the angle BCA. Why are there not two solutions?

Also construct and measure the perpendicular from A on BC and calculate the area of the triangle ABC.

3. State and prove the geometrical theorem corresponding to the algebraic formula $(a-b)^2 = a^2 - 2ab + b^2$.

In the figure AH, AD, BG are squares. Prove that the area of the pentagon $GMKLC$ is equal to the rectangle $AC{\cdot}CB$ together with half the sum of the squares on AC and CF.



4. Construct a cyclic quadrilateral being given that $AB = 4$ cm., $BC = 6$ cm., $A = 95°$, $B = 110°$.

5. From a point outside a circle two straight lines are drawn, one of which cuts the circle and the other meets it. What is the condition that the meeting line is a tangent? Give a proof.

In a triangle ABC angle $A = 80°$ and angle $C = 40°$. At A a line AD is drawn making angle $CAD = 60°$ and meeting CB at D. Prove that $CA^2 = CB, CD$.

6. A circle is described about a triangle ABC, and D is the middle point of the arc BC, being on the opposite side of the chord BC to A. If I is the centre of the inscribed circle, prove that $DI = DB = DC$.

No. 38

1 Show how to bisect a reflex angle

Two lines AB, AC meet at A Prove that the bisectors of the two angles at A (the reflex and the ordinary) are in the same straight line

2 Prove that the point of concurrence of the medians of a triangle is one third of the way from the bisected side to the opposite vertex

The side AD of a parallelogram $ABCD$ is bisected at E and CE meets the diagonal BD at F Prove that DF is one third of DB

3 A church A is 3 miles west of another church B A school is to be built so as not to be more than 2 miles from either church Draw a plan and mark the area in which the school may be built If it is as far as possible from both churches calculate its distance from the straight road AB, ssing by both churches
4

poin Prove that the line joining a point P to the centre O of a
Th bisects the angle between the tangents drawn to the
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on th e points A, B C are taken on the circumference of a
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5 Pro es in the angle between the tangents at A and C
intersecti

Perpen quadrilateral $ABCD$ is inscribed in a circle of radius
opposite s A is the tangent at A Construct the figure being
the line jo angle $EAD = 25°$ angle $ACB = 35°$, and angle
CK is perpe Calculate the angle between the diagonals and
accuracy of your figure
6 Construc
angle $BAC =$ point P inside a triangle ABC, perpendiculars
section of the re let fall on the sides BC, CA, AB respectively
$+ BD^2 + CE^2 = AE^2 + BF^2 + CD^2$
at the circles described about the triangles
meet in a point

No. 39

1. In a certain town the railway station A is 200 yds. east of a picture palace B; a church C is north-east of B and 250 yds. from A. Draw a plan showing the relative positions of A, B and C, and find the site for a house which is to be equidistant from A, B, and C.

2. In a certain pentagon three of the angles are each double each of the other two angles, find the number of degrees in each angle.

Write out in full the general enunciation you have used.

3. At the middle point O of a straight line AB, a perpendicular OC is drawn equal to OA; AC and BC are joined From any point P in OB a perpendicular is drawn to AB, meeting BC at D. DE is drawn parallel to PO to meet OC at E and line AD is drawn. Use this figure to prove—

$$AP^2 + PB^2 = 2AO^2 + 2OP^2.$$

4. Show that four circles can be drawn to touch three straight lines which are not concurrent, and no two of which are parallel.

Show that any one of the centres is the orthocentre of triangle formed by the other three centres.

5. In a circle of radius 3·5 cm. inscribe a triangle ABC having $A = 48°$, $B = 62°$. If the minor arc BC, CA, AB are bisected at A', B', C' respectively, measure and calculate the sizes of the angles A', B', C'.

6. ABC is an equilateral triangle and P a point inside it, on AP an equilateral triangle APQ is described, P and Q being on opposite sides of AC. BP, CQ are produced to meet at X. Prove that A, P, Q, X lie on a circle, and that the line joining the circumcentres of the triangles ABC, APQ is at right angles to AX.

No. 40

1. The bisectors of the opposite angles A and C of a parallelogram $ABCD$ meet the diagonal BD at E and F respectively, prove that AE is parallel to CF

2 The medians of a triangle ABC intersect at a point G Prove that the triangles BGC, CGA, AGB are equal in area

3 Draw, using a protractor, a regular pentagon $ABCDE$, and draw an isosceles triangle equal to it in area having D as a vertex Prove your construction to be correct

4 Two concentric circles are drawn, points A, B are taken on the inner circle and points C, D on the outer so that the angle OAB is equal to the angle OCD Prove that AC is equal to BD

5 Perpendiculars AD, BE, CF are drawn from the vertices of a triangle ABC to the opposite sides, P is the middle point of the side BC Prove that D, E, F, P lie on a circle

6 Prove that the perpendicular bisector of the chord of a circle passes through the centre

Two chords AB, CD of a circle with centre O intersect at right angles, the chords AD, BC are bisected at P and Q respectively Prove that $OP = CQ$

No. 41

1. $ABQP$ are four points in order on a straight line such that $AB = PQ$; ABC, PQR are equal angles, and $BC = QR$. Prove that $AR = PC$.

2. On a certain public common three paths intersect so as to form a triangle ABC. It is found that $AB = 120$ yds., the angle $BAC = 48°$, and the angles ABC and ACB are equal. There is a drinking-fountain at D equidistant from the three paths, but B and D are on opposite sides of AC. Draw a plan on the scale of 1 cm. = 30 yds., and explain your construction.

3. Prove that triangles on equal bases and of the same altitude are equal in area.

In a quadrilateral $ABCD$ the diagonals intersect at E; EB is produced to F so that $EF = DB$ and EC is produced to G, so that $EG = AC$. Prove that the triangle EFG is equal to the quadrilateral $ABCD$.

4. If two circles cut, prove that the common chord is bisected by the line joining the centres, produced if necessary.

A and B, distant 6 cm., are the centres of two circles which have a common chord PQ, 2·5 cm. long What are the loci of P and Q? Draw the two circles, being given that the tangents from B to the other circle are each of length 3·5 cm.

5. Perpendiculars AX, BY, CZ are let fall from the vertices of a triangle ABC to the opposite sides. Prove that AX bisects the angle YXZ.

6. If two chords of a circle intersect, prove that the rectangles contained by their segments are equal.

A triangle ABC, with angle $A = 60°$, is inscribed in a circle. A point P is taken on the circumference between A and B and a line $PQRS$ is drawn, cutting AB in Q, AC in R and the circle again in S, so that the angle AQR is also $60°$. Prove that the difference between AB and AC is equal to the difference between PR and QS.

No 42

1 Construct a triangle ABC in which $AB = BC$, $B = 80°$ and BD the perpendicular from B to $AC = 1\ 5$ in

Produce AB to E making $BE = AB$ Join CE Prove that $AE^2 = AC^2 + CE^2$ and verify by measurement and calculation

2 Four points A B C D are taken in that order on a straight line so that $AC = BD$ EF is any straight line parallel to AD Prove that the quadrilaterals $ACFE$ and $BDFE$ are equal in area

3 Prove that in an acute angled triangle the square on any side is less than the sum of the squares on the other two sides by twice the rectangle contained by either of those sides and the projection of the other on it

If the perpendicular from A on the opposite side BC meets it at a point D such that $BD = 2DC$ prove that $AB^2 = AC^2 + BC\ CD$

4 From a point P inside a circle three equal lines PA PB PC are drawn to the circumference show that P must be the centre of the circle

√5 Describe a triangle ABC in which $BC = 2\ 3$ in $B = 40°$ and the radius of the inscribed circle is 1 in State the steps of your construction

6 Explain how to draw two common tangents to two intersecting circles

Prove that the common chord when produced bisects the portion of either common tangent between the points of contact

No. 43

1. A point P is taken inside a triangle ABC, prove that $PB + PC < AB + AC$, and that $\angle BPC > \angle BAC$. Enunciate the converse propositions and say whether they are true.

2. Prove from the definition of a parallelogram that its diagonals bisect one another.

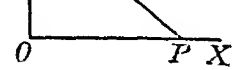
Through the point C a line is drawn parallel to the opposite side AB of a triangle ABC, and meeting the bisector of the angle BAC at D. If AB is not equal to AC, prove that BD is not parallel to AC.

3. Show, by drawing geometrical figures, that—

(i) $(x + 5)(x + 3) = x^2 + 8x + 15$.

(ii) $(x - 5)(x + 3) = x^2 - 2x - 15$.

4. A straight rod PQ slides so that P moves along OX and Q along OY; draw the locus traced out by the middle point R, taking twice the dimensions of the figure.



5. If two chords AB, CD of a circle are parallel, prove that $AC = BD$.

Two circles touch externally at P and a line cuts one of the circles at A, B, and the other at C, D; AP and BP are produced to meet the second circle at E and F respectively. Prove that EF is parallel to AD and that the angle EPC is equal to the angle BPD.

6. Construct a triangle ABC in which AD, the perpendicular from A to BC, is 3 in., AG (G being the intersection of the medians) is 2·4 in., and the radius of the circumcircle is 2 in.

No 44

1 An equilateral triangle CDE is described on the side CD of a square $ABCD$, the diagonals cut DE, CE at F and G respectively Prove that $EF = EG$

2 State the various ways you know of proving that two lines are parallel

In the figure of Ques 1 prove that FG is parallel to DC

3 State how to draw a line the length of which is exactly $\sqrt{21}$ in and prove your construction to be correct

4 Prove that the greatest line that can be drawn from a point outside a circle to a point on the circumference is that which passes through the centre

Two points A and B are fixed and are outside a given circle Find a point P on the circle so that $PA^2 + PB^2$ has the greatest possible value

5 Circles are inscribed in each of the four triangles into which a parallelogram is divided by its diagonals Prove that the quadrilateral formed by joining the centres of these circles is a rhombus

6 Unequal perpendiculars AP, BQ are drawn at the extremities A and B of a straight line AB and a circle is described on PQ as diameter cutting BQ the larger of the two perpendiculars at R Prove that $AP = BR$

If this circle cuts AB at X and Y prove that $AX = BY$, and if $AP = 1$ $AB = a$ and $BQ = b$ prove that the lengths of AX and AY are the roots of the equation $x^2 - ax + b = 0$

No. 45

1. Prove that the middle point of any straight line which meets two parallel straight lines is equidistant from these lines.

Explain how to find, without measuring any line or angle, a point which is equidistant from each pair of two pairs of parallel straight lines.

2. In a quadrilateral $ABCD$ the diagonal BD bisects the quadrilateral and is of length 5 cm.; $AB = 3\cdot9$ cm., $CD = 3$ cm., and angle $BAD = 115°$. Construct the quadrilateral and measure BC.

3. Prove that the perpendiculars from the vertices of a triangle to the opposite sides are concurrent.

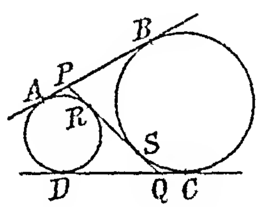
From the vertices B, C of a triangle ABC perpendiculars BX, CY are let fall on the opposite sides. Prove that—

(i) $AX.AC = AY.AB$;
(ii) $BY.BA + CX.CA = BC^2$.

4. Draw two non-parallel straight lines which do not meet on your paper. Explain how to draw a line which would, if produced, bisect the angle between the original two lines.

5. Show how four common tangents may be drawn to two non-intersecting circles.

The figure shows two circles with two direct common tangents and one transverse. Prove $AB = PQ$.



6. AB is the diameter of a circle. Through A two chords AP, AQ are drawn and produced to meet the tangent at B in R and S respectively. Prove that, whether AP, AQ are on the same side of AB or on opposite sides, the four points P, Q, R, S are concyclic.

No 46

1 OX, OY are two mirrors, the reflecting faces containing an angle of 40° P is a luminous point 3 ft from OX and 2 ft from OY Draw a diagram on the scale 1 cm = 1 ft showing P and four images P_1 P_2 P_3 P_4 Prove that the points P, P_1 P_2 P_3 P_4 lie on a circle

2 If a parallelogram and a triangle are on the same base and between the same parallels prove that the area of the parallelogram is double that of the triangle

$ABCD$ is a parallelogram in which AB is greater than AD A point H is taken inside the triangle ACD Prove that—

$$\triangle AHC = \triangle AHB - \triangle AHD$$

3 Draw figures to illustrate the algebraical formula—

(i) $(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2$

(ii) $(a+b)^2 - (a-b)^2 = 4ab$

4 In any circle prove that equal chords are the same distance from the centre

Draw a circle radius 2 in and take a point P 3 2 in from the centre Show with proof how to draw a line PAB cutting the circle at A and B so that $AB = 2$ 9 in

5 If at a point on a circle a tangent and chord are drawn either angle between the tangent and chord is equal to the angle in the segment on the other side of the chord

From a fixed point O tangent OA OB are drawn to a fixed circle any secant OPQ is drawn and a chord BR parallel to the secant If AR meets PQ at X prove that X lies on a fixed circle Hence show that X is the mid point of PQ

6 AB is a chord of a circle and C is any point outside the circle Show with proof, how to draw a secant CDE so that DE is bisected by AB

No. 47

1. Prove that two quadrilaterals $ABCD$, $PQRS$ are congruent if $AB = PQ$, $BC = QR$, $CD = RS$, and $\angle ABC = \angle PQR$, $\angle BCD = \angle QRS$.

2. Prove that the angles at the base of an isosceles triangle are equal.

In a triangle ABC the sides AB, AC are equal and the base BC is produced through C to any point D. From D, DE is let fall perpendicular to AB and DF perpendicular to AC produced. Prove that BD bisects the angle EDF.

3. If a number of parallel lines divide any transversal into equal parts, prove that any other transversal will also be divided into equal parts.

Draw a line AB of length 3·7 in., divide it into two parts AP, BP such that $3AP = 4BP$. Give a proof.

4. Explain how to divide a straight line AB into two parts at C so that rectangle $AB.BC$ is equal to the square on AC.

If $ABDE$ is the rectangle $AB.BC$, prove that $AD^2 = 3\ AC^2$.

5. Prove that the line from the centre at right angles to a chord bisects the chord.

Two circles intersect at A; show, with proof, how to draw a line PAQ meeting the circles at PQ so that PQ is bisected at A.

6. Explain how a regular pentagon may be drawn without using a protractor.

No. 48

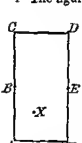
1. Two equilateral triangles BAC, QAP have a common vertex A, show that the sides of the triangle QPC are respectively equal to QA, QB, QC

2 What is the locus of points equidistant from the arms of a given angle?

A point Q is taken in the base YZ produced of a triangle XYZ in which $XY = XZ$, perpendiculars are let fall from Q on XY, XZ, produced if necessary Show that the difference between the perpendicular is the same whatever point Q is taken in the produced base

3 Construct a triangle ABC, being given that median $AD = 4 \cdot 6$ cm, median $BE = 3 \cdot 5$ cm, and side $AB = 4$ cm

4 The figure represents a billiards table 12 ft $\times$ 6 ft, with pockets at A, B, C, D, E, F, a ball is placed at X, 2 ft from AC and 3 ft from AF. Regarding the ball and pockets as points, and assuming that the paths of the ball before and after striking a side are equally inclined to the side draw a scale figure to find (i) a point P in AC such that the ball after striking at P may go into the pocket D, and (ii) a point Q in AC such that the ball after striking at Q may hit the top cushion CD and then go into the pocket F



5 The sides BC, CA, AB of a triangle ABC are respectively 6 in, 9 in, 10 in. long The inscribed circle of the triangle touches these sides at D, E, F respectively Calculate the lengths of BD, CE, AF If the inscribed circle touching BC, not produced, touches AB, produced at P, calculate the length of AP

6 From a point P on the circumcircle of a triangle ABC perpendiculars PH, PK, PL are let fall on the sides BC, CA, AB, produced if necessary Prove that H, K, L lie on a straight line

No. 49

1. A pirate buried some treasure at a place C, near two trees A and B, B being 57 yds. W. of A. He made a note that the angle CAB was $50°$ N. of AB and BC was equal to 47 yds. On returning some years later he made correct measurements, but failed to find the treasure. After consideration he made a second attempt and found the treasure. Draw a figure to the scale 1 cm. = 10 yds. to explain his mistake. What was the distance between the two holes ?

2. The middle point E of the side AD of a parallelogram is joined to B, and BE cuts the diagonal AC at F. Prove that the triangle BCF is one-third of the parallelogram.

3. Four points A, B, C, D are taken in this order on a straight line ; prove (i) by means of a figure, (ii) algebraically, that $AD^2 + BC^2 = AC^2 + BD^2 + 2\,AB\,.\,CD$.

4. Prove that the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the diagonals together with four times the square on the line joining the middle points of the diagonals.

5. Show that the acute angle between a tangent and a chord through the point of contact is half the angle at the centre subtended by the chord.

Two circles touch internally at A and a chord ABC meets the circles at B and C. The tangent at B meets the outer circle at D and E ; prove that the tangents at B and C are parallel and that the arc CD is equal to the arc CE.

6. Construct a triangle ABC in which the radius of the inscribed circle is 1 in., the distance of A from the centre of that circle is 2·5 in., the side BC is 4 in.

No 50

1 A point P is taken inside a triangle ABC prove that $AP + BP + CP$ is less than the sum of the sides of the triangle ABC but greater than half the sum of the sides

2 Prove that the bisector of the angles of any quadrilateral enclose a cyclic quadrilateral

If this cyclic quadrilateral is a rectangle, show that the original quadrilateral must be a parallelogram

3 State and prove a rule for finding the area of a trapezium

Draw a trapezium $ABCD$ in which AB is parallel to CD and in which $AB = 3\ 4$ cm , $BC = 4$ cm , $CD = 6$ cm , $DA = 4\ 3$ cm Use your rule to find the area of the trapezium

4 Through a point P 3 cm from the centre O of a circle with radius 5 cm chords are drawn Show that the locus of the middle points of these chords is a circle Draw the locus State with proof, how to draw the longest and shortest of these chords and calculate their lengths

5 Enunciate the proposition concerning the tangent and a secant drawn to a circle from an external point

Explain, with proof, how to draw a circle to pass through two given points and to touch a given circle

6 A triangle ABC being given, explain how to draw three circles with their centres at A, B, and C, such that each circle touches the other two

If these circles touch at D E, and F and if the $\angle D = 30°$ and $\angle E = 70°$ calculate the angles of the triangle ABC

No. 51

1. Equilateral triangles PBC, QCA, RAB are described on the sides of a triangle ABC, all falling outside the triangle. Prove that $PA = QB = RC$.

2. In any triangle prove that the sum of the squares on two sides is equal to twice the square on half the third side, together with twice the square on the median bisecting the third side.

Prove that the sum of the squares on the diagonals of a parallelogram is equal to the sum of the squares on the four sides.

3. Prove that an angle at the centre of a circle is double any angle at the circumference standing on the same arc.

AB, AC are equal chords of a circle with centre O, and AD is another chord cutting BC at E. Prove that the angle AEC is one half of one of the angles AOD.

4. State the construction for inscribing a circle in a triangle and prove it to be correct.

Find, by drawing, the radius of the biggest sphere that can be covered by a cone of height, 8 in., and with a base of radius of 6 in.

5. Prove that a straight line parallel to one side of a triangle divides the other two sides in the same ratio.

A point P is taken on the side AB of a triangle ABC, such that $AP = 2PB$. Through P, PQ is drawn parallel to BC, meeting AC at Q, and PR is drawn parallel to BQ, meeting AC at R. Find the ratio $AR : RC$.

6. In a triangle ABC, the angle $C = 90°$ and the angle $B = 30°$; CB is produced to D so that $BD = BA$. From this figure calculate tan 15°, sin 15°, each to three decimal places.

No. 52

1 Define a parallelogram, and prove that if both pairs of opposite sides of a quadrilateral are equal the quadrilateral is a parallelogram

P, Q, R S, are the middle points of the sides of an irregular quadrilateral, prove that PR, QS bisect one another

2 *Prove that all points equidistant from two fixed points* A and B lie on a certain fixed straight line

Draw a triangle ABC having $AB = 6\ 7$ cm, $BC = 7\ 8$ cm $CA = 8\ 9$ cm, on AB construct a triangle VAB, such that $VA = VB = VC$

3 On two sides, AB, AC of a triangle as diameters circles are described Prove that D, the other point of intersection lies on BC or BC produced

4 Explain what is meant by *similar figures* If two triangles have their corresponding sides in the same ratio, prove that they are equiangular

5 If four lines are in proportion give a geometrical proof that the rectangle contained by the means is equal to the rectangle contained by the extremes

From a point P, two tangents PA, PB are drawn to a circle with centre O, and PO cuts AB at C Prove that $PC \cdot PO = PA^2$

6 A point P is taken distant a inches from the centre O of a circle with radius r, from a point Q on the circumference QN is let fall perpendicular to PO, produced if necessary If the angle QON is $x°$, find expressions for the lengths of PN, QN, PQ

From the expression for PQ deduce that PQ is greatest or least when it lies along the line PO

No. 53

1. Two isosceles triangles PAB, QAB are on the same side of the same base (i.e. unequal side) AB. Prove that PQ when produced bisects AB.

2. If two triangles have two sides of one equal respectively to two sides of the other, but the angle contained by one pair greater than the angle contained by the other pair, then the third side of the triangle with the greater included angle is greater than the third side of the other.

In a quadrilateral $ABCD$ the opposite sides AD, BC are equal, but the angle DAB is greater than the angle ABC. Prove that the angle BCD is greater than the angle ADC.

3. Prove that a triangle is right-angled if the square on one side is equal to the sum of the squares on the other two.

Calculate the area of a quadrilateral $ABCD$ in which $AB = 6$ cm., $BC = 2{\cdot}5$ cm., $CD = 3{\cdot}3$ cm., $DA = 5{\cdot}6$ cm., angle $ABC = 90°$.

4. Prove that the two tangents drawn to a circle from an external point are equal.

A quadrilateral is such that the sum of one pair of opposite sides is equal to the sum of the other pair of opposite sides. Prove that the bisectors of the angles of the quadrilateral are concurrent.

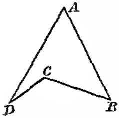
5. Prove that the bisector of an angle of a triangle divides the opposite side proportionally to the sides containing the angle.

The middle points of the sides BC, CA, AB of a triangle are P, Q, R respectively, and X is a point in QR such that $QX : XR = AB : AC$. Prove that PX bisects the angle RPQ.

6. From a point P a tangent PT is drawn to a circle of which TR is a diameter, and PR cuts the circle at Q. Calculate the length of PQ, being given $TR = 9{\cdot}6$ and angle TPQ is $63° \, 15'$.

No. 54

1. It is required to make an exact copy of the quadrilateral *ABCD* by measuring one length and as many angles as are *necessary*. Record your measurements of the length (to the nearest tenth of an inch) and angles, and explain why they are sufficient to make a copy of the figure.



2. If in a triangle one side is greater than another, prove that the angle opposite the greater side is greater than the angle opposite the less.

Construct a triangle ABC so that $BC = 3{\cdot}4$ in., angle $B = 42°$, and AB exceeds AC by $1{\cdot}6$ in. Measure AC.

3. Construct, without any calculation, a rectangle with one side $1{\cdot}7$ in. equal in area to a rectangle with sides $3{\cdot}2$ in. and $1{\cdot}2$ in. State and prove your construction.

4. If two chords of a circle intersect internally or externally, prove that the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.

Circles are drawn to pass through two fixed points A and B. What are the respective loci of—

(i) The centres of these circles?

(ii) The points of contact of the tangents drawn to these circles from a fixed point C in AB produced?

5. Explain how to draw a figure similar to one rectilineal figure A, and equal to another rectilineal figure B.

Construct geometrically an isosceles triangle whose sides are in the ratio $3 : 3 : 2$ equal in area to a square of side $2{\cdot}5$ in.

6. (i) Construct, without using tables, an angle A, being given $\sin\frac{A}{2} = {\cdot}8$, and from the figure find the values of $\sin A$ and $\cos A$.

Check your result by using protractor and tables.

(ii) State and prove a formula giving the cosine of an angle of a triangle which is not right angled in terms of the sides.

No. 55

1. State and prove a formula for obtaining the size of an internal angle of a regular polygon of n sides.

A square $ABCD$ and a regular pentagon $ABPQR$ are described on opposite sides of a line CD. Find, by calculation, the angles of the pentagon $PQRDC$.

2. Prove that the opposite sides of a parallelogram are equal.

In a parallelogram $ABCD$ the side AB is greater than the side AD. Prove that the point E, where the bisector of the angle A cuts the side DC, is between C and D.

3. Prove (using, if you like, the properties of similar triangles) that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

AB and PQ are chords in two circles with centres C and O respectively, the former circle having the longer radius; and AB, PQ are at equal distances from C and O respectively. Prove that AB is greater than PQ.

4. $ABCD$ is a cyclic quadrilateral in which AD is equal to CD; I is the centre of the inscribed circle of the triangle ABC. Prove that D is the centre of the circumcircle of the triangle AIC.

5. The bisector of the angle A of a triangle ABC meets BC at D; I is the centre of the inscribed circle and E of the described circle touching BC, not produced. Prove that the rectangle $BI \, . \, DE$ is equal to the rectangle $CE \, . \, DB$.

6. The ground at the foot A of a vertical wall AB slopes downwards at an angle 32° with the horizontal. A ladder 18 ft. long, placed at a point C on the ground, just reaches B, the top of the wall. If AC is equal 5 ft. 8 in., calculate the height of the wall and the inclination of the ladder to the vertical.

No. 56

1 If two straight lines AB CD intersect at O, and the opposite vertical angles AOC, BOD are bisected by OP, OQ respectively, prove that PO and OQ are the same straight line

If X, a point on the bisector of the angle AOD, is joined to Y, any point in OP, prove that the circle having XY as diameter will pass through O

2 On the same side of a line XO, draw angle XOC equal to 50°, and another angle XOY equal to 117°, and make OC = 1 7 in Draw a straight line through C, cutting OX at A and OY at B, such that $AC = CB$ State and prove your construction

3 Prove that the opposite angles of a cyclic quadrilateral are equal to two right angles (Try to prove this without using any property of a circle, except that the radii are equal)

The opposite sides AB, DC of a quadrilateral $ABCD$ meet at P when produced and the rectangle PA PB is equal to the rectangle PC PD Prove that the angles DAC, DBC are equal

4 The sides of a triangle ABC are a b c, show that the radius of the inscribed circle is equal to twice the area of the triangle divided by $(a+b+c)$

Calculate the length of the radius of the inscribed circle of a triangle whose sides are 73 cm 55 cm 48 cm

5 Prove that two triangles are similar if they have two sides of the one proportional to two sides of the other, and the angles included by those sides equal

Tangents PA, PB are drawn to a circle with centre O, and PO cuts AB at C, XY is any other chord through C Prove that—

(i) Rectangle PC $CO = CA^2$

(ii) Triangles PCX, OCY are similar

(iii) P, X, O, Y lie on a circle

(iv) PO bisects the angle XPY

6 A road runs east from A and on the north side of the road is a forest A man walks 6 miles from A along the road, and then turns northward through an angle of 70°, and walks 5 miles into the forest Find by calculation, how far he is then from A and from the road In what direction must he walk to return direct to A ?

No. 57

1. What do you know about the diagonals of (i) a parallelogram, (ii) a rhombus, (iii) a rectangle, (iv) a square, (v) a cyclic quadrilateral.

In a quadrilateral $ABCD$ the diagonals AC and BD are equal, and the angles at B and C are both right angles. Prove that the quadrilateral is a rectangle.

2. The diagonals AC, BD of a quadrilateral are at right angles and intersect at O; show by a figure that the area of the quadrilateral is half the rectangle to AC, BD.

3. Through a point P in the diagonal AC of a parallelogram $ABCD$ are two lines HPK, XPY, each parallel to two sides of the parallelogram, meeting AB at Y, BC at K, CD at X, DA at H. Prove that—

(i) The parallelograms $DHPX$, $BKPY$ are equal in area;

(ii) The parallelograms $AYPH$, $CXPK$ are similar to one another and to $ABCD$.

4. Draw, and explain, figures to illustrate the algebraical formula—

(i) $(a - b)^2 = a^2 - 2\,ab + b^2$;

(ii) $(a + b)^2 - (a - b)^2 = 4\,ab$.

5. A quadrilateral $ABCD$ is such that a circle can be drawn to touch all four sides; prove that—

$$AB + CD = AD + BC.$$

If P, Q, R, S are the respective points of contact with the sides AB, BC, CD, DA, express the angles of the quadrilateral $PQRS$ in terms of the angles of the quadrilateral $ABCD$.

6. A quadrilateral $ABCD$ is determined by the measurements $AB = 3$ in., $BC = 3{\cdot}8$ in., $CD = 2{\cdot}9$ in., $BAD = 106°$, $BCD = 90°$.

Construct the quadrilateral; find its area by constructing a triangle equal to it in area.

What are the corresponding measurements of a similar quadrilateral $A'\,B'\,C'\,D'$ of which the area is $\frac{4}{9}$ the area of $ABCD$?

No. 58

1 Prove that, if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, the two triangles are equal in every respect

P and Q are two points on the same side of a straight line XY, but at unequal distances Find a point O in XY such that PO, QO may be equally inclined to XY

2 Prove that in any triangle the difference of the squares on two sides is equal to twice the rectangle contained by the third side and the projection on it of the median bisecting that side

3 Construct a parallelogram $ABCD$ of area 3 sq in, such that $AB = DC = 1\ 5$ in and the perpendicular distance between AD and BC is 1 25 in

Explain your construction

4 Define a tangent to a circle Use your definition to prove that the tangent to a circle at any point on it is perpendicular to the radius drawn to that point

Two circles touch at A, from P any point on the common tangent at A, tangents PQ, PR are drawn to the circles Show, with proof, how to draw a circle to touch the two circles at Q and R respectively

5 State the ratio property of the bisector of an angle of a triangle

Two points A and B are 3 cm apart, draw the locus of a point P which moves so that $PA : PB = 2 : 1$

6 A man, whose eyes are 5 ft 6 in above the ground, stands at a horizontal distance 30 ft from the foot of a vertical tower 55 ft high Find, as nearly as your tables permit, the angle subtended by the tower at the man's eye

No. 59

1. Define a parallelogram ; from your definition prove that either diagonal bisects the parallelogram.

Any point E is taken on the diagonal AC of a parallelogram $ABCD$; prove that the triangles BCE, DCE are equal in area.

2. Prove that triangles on the same base and between the same parallels are equal in area.

Prove that of all such triangles the isosceles triangle has the smallest perimeter.

3. If a straight line is divided equally and also unequally, the rectangle contained by the unequal parts is equal to the difference of the squares on half the line and the line between the points of section.

Explain how to divide a given finite straight line so that the rectangle contained by the parts may have (i) the maximum area, (ii) the maximum perimeter.

4. Of two chords of a circle drawn through a given point, prove that the one farther from the centre is less than the other.

A point being given within a given circle, explain how to draw a chord so that the rectangle contained by the segments shall have (i) the minimum area, (ii) the minimum perimeter.

5. Divide, by a geometrical construction, a line AB, 3 in. long, at a point P, so that $AP^2 = 2\,BP^2$.

Do the same by algebra and thus check your result.

6. If A is an acute angle of a triangle with hypotenuse c, show that the area is $\frac{1}{2} c^2 \sin A \cos A$. Deduce the maximum area for a right-angled triangle with hypotenuse c and verify by pure geometry.

No. 60

1 Without using a protractor, construct an angle of $52\frac{1}{2}°$ Give proof that your construction is correct

2 If two sides of a triangle are equal, prove that two of the angles are equal

On the equal sides AB, AC of a triangle, squares $ACDE$, $ABHK$ are described Prove that KE is parallel to BC.

3 Explain how to make a square equal to a given rectangle

In a straight line AB 3 5 in long, find, by a geometrical construction, a point P such that $AP^2 - PB^2$ may equal the square on a line 2 in long

4 Prove that the perpendicular from the centre of a circle to any chord bisects the chord

From the extremity A of a diameter AB of a circle, chords are drawn What is the locus of the middle points of these chords?

5 Two circles touch one another and also touch a straight line at points A and B Prove that AB is a mean proportional between the diameters of the circles

6 A rod OP of length a inches is turned about O by another rod AP of length b inches of which the end A is compelled to move to and fro along a portion of the line OX Show that, when the angle AOP is $x°$, $OA = a \cos x + \sqrt{b^2 - a^2 \sin^2 x}$ If $a = 7$, $b = 12$, find the length of that part of OX along which A moves

Also find the greatest value of the angle OAP

No. 61

1. Prove, without using proportion, that the straight line drawn through the middle point of one side parallel to another side bisects the third side.

Prove that the line joining E and F, the middle points respectively of AD and BC, the non-parallel sides of a trapezium, is parallel to AB and CD, and equal to half their sum.

2. Prove that the three medians of a triangle are concurrent.

Construct a triangle, being given that the lengths of the three medians are 2·7, 3·4, 1·9 respectively.

3. Prove that in equal circles equal chords subtend equal angles at points in the circumference.

Two equal circles cut at A and B; with A as centre a circle is described cutting the two circles at C and D on the same side of AB. Prove that BCD is a straight line.

4. The inscribed circle of a triangle ABC touches the sides CA, AB, at E, F respectively. Prove that AE or AF equals half the perimeter of the triangle diminished by the side BC.

Draw a triangle ABC in which $BC = 7·3$ cm., $CA = 6·9$ cm., $AB = 5·1$ cm. Now with centres A, B, and C draw three circles, each of which touches the other two.

5. If a perpendicular be let fall from the right angle of a right-angled triangle upon the hypotenuse prove that the triangle is divided into triangles similar to the whole triangle and to one another.

From a point P a tangent PD is drawn to a given circle; from D a perpendicular DC is drawn to the diameter AB, which, when produced, passes through P. Prove that $AC : CB = AP : PB$.

6. In a triangle ABC, $AB = 13$ in., $AC = 4$ in., and AD, the perpendicular on BC, $= 3·2$ in. Calculate the magnitudes of the angles ABC and ACB and the length of BC.

No. 62

1 Prove that the diagonals of an equal-sided quadrilateral bisect one another at right angles

$ABCD$ is a square , explain, with proof, how to draw a line parallel to the diagonal BD and cutting AB at P, and AD at Q, such that $PQ = AB$

2 Construct a quadrilateral $ABCD$, having $AB = 4$ in , $BC = 2$ in , $CD = 3\ 5$ in , angle $BAD = 60°$, angle $ABC = 110°$, and the angle ADC acute Bisect the straight quadrilateral by a straight line drawn through B State and prove your construction

3 Give conditions that four points may be concyclic

If four circles are drawn so that each circle touches two only of the other three, determine whether the four points of contact are concyclic

4 The perpendiculars AD, BE, CF, drawn from the vertices of a triangle to the opposite sides intersect at K and meet the opposite sides at D E, F Prove that $AK\ KD = BK\ KE = CK\ KF$

5 If in a triangle ABC the external bisector of the angle A meets BC, produced at D, prove that $BA\ \ AC = BD\ \ DC$

Two points A and B are at a distance 4 cm Construct the locus of a point which moves so that $AP\ \ PB = 2\ \ 3$

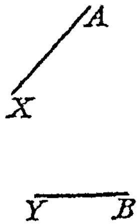
6 A path AX is drawn from the foot A of a column AB, 75 ft high, on the top of which is a statue BC, 16 ft high State how to find the point P in AX such that the angle BPC is as large as possible Find the size of that maximum angle by trigonometrical calculation

No. 63

1. At the extremity A of a line AB, which cannot be produced through A, erect, without using protractor or set square, a line AC perpendicular to AB. Prove your construction.

2. Take two points A and B in the left-hand edge of your paper and draw two diverging lines AX and BY. Construct the line CZ, which would, if produced off the paper, bisect the angle between XA and YB if they also were produced off the paper to meet. State and prove your construction.

3. In the Figure, A and B are the opposite corners of a parallelogram $PAQB$, but AX and BY cannot be produced to meet; if they could, P would be their point of intersection. Construct as much as you can of the diagonal PQ. State and prove your construction.



4. In the Figure of Question 3, suppose that AX and BY are obtained by producing two adjacent sides of a parallelogram, the lengths of the sides being 2 in. and 3 in. respectively. Construct the line PQ, which is the prolongation of the diagonal through the intersection of those sides. State and prove your construction.

5. Draw an arc AB of a circle. Without finding the centre draw the tangents at A and B.

Take any point P outside the segment but within the triangle formed by the chord AB and the tangents at A and B; show how the tangent from P may be drawn without finding the centre.

6. A sphere of radius 6 in. rests on a hollow cylinder of radius 4 in.; find by drawing and by calculation, the distance of the centre of the sphere from the centre of the top rim of the cylinder.

No 64

1 Prove that if two angles of a triangle are equal two sides also are equal.

A point P in the hypotenuse AB of a right angled triangle ABC is such that the angle $PAC =$ angle PCA prove that $AP = PB$

2 AB is a line 3 5 cm long draw the complete locus of a point P which moves so that the area of the triangle PAB is 3 5 sq cm and that no side of the triangle is greater than AB

3 Prove that the sum of the squares on two sides of a triangle is equal to twice the square on half the third side together with twice the square on the median bisecting the third side

In a quadrilateral $ABCD$ E is the middle point of the diagonal AC and F of BD Prove that the sum of the squares on the sides of the quadrilateral is less than the sum of the squares on the diagonals by four times the square on EF

4 If the opposite sides of a quadrilateral are together equal to half the perimeter of the quadrilateral prove that a circle can be described to touch all the sides of the quadrilateral

5 Explain how to draw a rectilineal figure similar to a given rectilineal figure and with its area p times the area of the given figure

As an example draw an equilateral triangle five times as large as the equilateral triangle with side 1 3 in

6 A cone of height 7 cm with vertex V has a circular base whose diameter AB is 4 7 cm Draw to scale the section of the cone made by a plane parallel to the base the area of the section being one third the area of the base

Find by calculation the size of the angle AVB

No. 65

1. Give the enunciations of three propositions which state that a straight line is greater than some other line.

Draw a triangle ACB having $AB = 2{\cdot}3$ in., $AC = 1{\cdot}7$ in., $BC = 1{\cdot}1$ in. Now show within what area a point P must fall in order that AP may be greater than $1{\cdot}7$ in., but BP less than $1{\cdot}1$ in.

2. Lines are drawn through the vertices of a triangle ABC parallel to the opposite sides, forming the triangle XYZ. Show that the area of XYZ is four times that of ABC.

Prove the medians of any triangle are also the medians of the triangle formed by joining the middle points of the sides.

3. Construct a parallelogram with one side $AB = 3{\cdot}2$ cm., the angle $ABC = 50°$, and the diagonal $BD = 5{\cdot}8$ cm.

4. Take any four points A, B, C, D in a straight line and show how to find a point O in the line AD so that the rectangle $OA \, . \, OC =$ rectangle $OB \, . \, OD$.

5. Prove that two similar rectilineal figures can be divided into pairs of similar triangles.

ABC is an equilateral triangle with side 2 in., XYZ is an equilateral triangle with side $1{\cdot}7$ in.; a point P is taken inside the triangle ABC so that $AP = 1{\cdot}3$ in. and PC bisects the angle C. Find a point O in the triangle XYZ such that it will divide XYZ into triangles similar to PAB, PBC, PCA.

6. A spherical balloon of radius 20 ft. subtends an angle $10° \, 18'$ at an observer's eye when the angle of elevation of the centre is $56° \, 41'$. What is the height of the centre above the horizontal level of the observer's eye?

No 66

1 State and prove the congruency enunciation which does not mention any angles

A triangle ABC is formed by three thin rods AB, BC, CA, the triangle is turned about B through an angle to the position $A'BC'$ What does the above mentioned enunciation tell you about the angles of the triangle $A'BC'$? Prove that one of the angles between AC and $A'C'$ is equal to the angle ABA'

2 Through the middle point D of the side BC of a triangle ABC a line is drawn making equal angles AXD, AYD, with AB, AC respectively Prove that $AX + AY = AB + AC$

Draw any angle PAQ and take a point D within the angle QAP Show how to construct a triangle ABC so that BC passes through D, meets AP, AQ at B and C respectively, and so that the triangle ABC has the smallest possible area

3 Define a tangent to a circle and from the definition prove that the perpendicular to any radius at its extremity is a tangent to the circle

What other properties of a tangent do you know?

What is the locus of points from which the tangents to given two intersecting circles are equal?

4 Draw a circle with radius 75 in about it describe, without using a protractor, a triangle having two of its angles 30° and 45° respectively

5 AD and PS are medians of two equiangular triangles ABC, PQR Prove that $AD : PS = BC : QR$

6 The circumcircle of an isosceles triangle ABC meets AD, the perpendicular bisector of BC, when produced at E Through D any chord PDQ is drawn, and EP, EQ cut BC, produced when necessary at R and S Prove that the rectangle $DR \cdot DS$ is constant

No. 67

1. Prove that the opposite sides and angles of a parallelogram are equal.

$ABCD$ is a parallelogram; AB is produced to a point E such that DE bisects BC. Prove that $AB = BE$.

2. Divide a straight line 3·8 in. long into two parts so that the rectangle contained by them may be of area 3 sq. in. State and prove your construction.

3. Prove that the perpendicular bisector of any chord of a circle must pass through the centre.

Perpendicular AD, BE are let fall from the vertices A and B of a triangle ABC to meet the opposite sides at D and E; K is the point of intersection of AD and BE. Prove that the line joining the mid-points of AB and CK bisects DE at right angles.

4. Given a square whose side is 1·5 in., describe a regular octagon by cutting off the four corners. Explain your construction.

5. A straight AB is divided internally at P and externally at Q in the same ratio and O is the middle point of AB.

Prove (i) $OP, OQ = OA^2$, (ii) $\frac{1}{AP} + \frac{1}{AQ} = \frac{2}{AB}$

6. Give a geometrical construction, with proof, for making an isosceles triangle with each of the base angles double of the vertical angle.

Deduce that $\sin 54° = \frac{\sqrt{5}+1}{4}$.

No 68

1 Prove that the sum of the interior angles of a polygon of n sides is equal to $180(n-2)$ degrees

In a pentagon $ABCDE$ the angles at A and B are each $110°$ and the angles at C and E are each $100°$ The alternate sides are produced to meet, forming a star shaped figure What is the sum of all its angles ? Calculate the magnitudes of the angles and verify that the sum is right

2 Show that if one diagonal of a quadrilateral divides it into two equal parts it need not be a parallelogram, but if both diagonals divide it into equal parts it must be a parallelogram

3 If two circles touch, prove that the centres and point of contact are in the same straight line

A circle is drawn touching a given circle whose centre is A and also a given straight line BC Show that the centre is equidistant from A and from one of two straight lines parallel to BC

4 Prove geometrically that, if a straight line AB be divided equally at O and unequally at X,

$$AX \cdot XB = AO^2 - OX^2$$

Divide a straight line 7 cm long into two parts so that the difference of the squares on these parts may be 6 sq cm

5 Without assuming any propositions about a circle prove that the angle in a semicircle is a right angle

Two points P and Q 1 in apart are the middle points of adjacent sides of an unknown rectangle Construct the loci of its vertices

6 Prove that if two similar triangles have their corresponding sides parallel the lines joining corresponding vertices meet in a point

In a given triangle ABC show to inscribe a square $PQRS$ so that P lies on AB Q on AC, R and S on BC

If x is the length of the side of this square and h the length of the perpendicular AD to the side BC, prove that $\frac{1}{x} - \frac{1}{h} = \frac{1}{BC}$

No. 69

1. Two equilateral triangles ABC, CDE are so placed that D is inside the triangle ABC and E on the opposite side of CA. Prove that $AE = BD$.

2. If a parallelogram and triangle are on the same base and between the same parallels, state, and prove, the relation between their areas.

On the side AB of a triangle ABC the square $ABDE$ is described. If the area of the triangle DBC is equal to the area of the square, prove that the triangle ABC is obtuse-angled, and that either the side AE of the square will, produced if necessary, bisect BC, or the side BD will cut the side AC in a point of trisection.

3. State the construction for drawing the internal common tangents of two non-intersecting circles.

Calculate their lengths when the radii are 3 in. and 4 in., and the centres are 8 in. apart.

4. If A, B, C are three points in a plane, prove that $AB^2 + AC^2 + 2AB, AC$ is greater than BC^2 unless A, B, C are collinear and A between B and C.

5. Two circles, $ABQX$, $ABPY$, cut at A and B. A, P, Q are in a straight line and B, X, Y are in a straight line. Prove that PY, QX are parallel.

6. OX, OY are two lines inclined at an angle 70°. A straight line AB, 1 in. long, slides so that A is always on OX and B on OY. Construct the locus of the centre of the circumcircle of the triangle AOB.

No. 70

1 If the line AD which bisects the base BC of a triangle ABC also bisects the angle BAC, prove that $AB = AC$

2 On the same base BC and on the same side of it are an isosceles triangle ABC and a triangle DBC equal to it in area Prove that the isosceles triangle has the smaller perimeter

3 Construct a trapezium $ABDC$ in which $AC = 2$ cm, $BD = 3$ cm, $CD = 2.6$ cm, and AC, BD are both perpendicular to AB

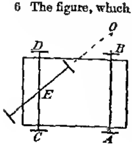
Find, by calculation, the length of AB and the area of $ABCD$

4 Two circles with centres A and B, 7 4 cm apart, and of radii 7 0 cm and 2 4 cm respectively, intersect at P and Q Prove that BP and BQ are tangents to the circle with centre A

5 If two chords of a circle intersect within the circle, prove that the rectangles contained by their segments are equal

Two chords of a circle, PA and PB, are inclined at an angle 60°, another chord CD cuts PA at Q and PB at R so that $PQ = QR$ Prove that the difference between PA and PB equals the difference between QC and RD

6 The figure, which is not drawn to scale shows the plan of four wheels of a cart The axles AB and CD are 4 ft in length and 5 ft 9 in apart In turning, the front axle is turned about, its mid point E through an angle of 8° 20, so that the cart turns about a point O Find by calculation the radii of the circles described by the wheels



No. 71

1. Draw two non-congruent triangles ABC, XYZ, having AB, BC equal to XY, YZ, each to each, and the angle BAC equal to the angle YXZ. State, and prove, the relation connecting the angles opposite the other pair of equal sides.

2. Prove that parallelograms on the same base and between the same parallels are equal in area. (The proof must be based on the fact that areas which can be made to coincide are equal.)

Construct a trapezium $ABCD$ having $AB = 4{\cdot}5$ cm., $BC = 3{\cdot}6$ cm., $CD = 5$ cm., $DA = 4$ cm., and AB parallel to DC. Make a rectangle equal to it in area.

3. By how much does the sum of the squares on the sides containing an acute angle exceed the square on the remaining side of the triangle? Prove your statement when the triangle is obtuse-angled.

4. ABC is a triangle with C obtuse, and D is a point such that BC produced bisects AD at right angles. Prove that BD is a tangent to the circumcircle of the triangle ABC if $BC = CA$; and that if BD cuts the circle at a point P between B and D then BC is greater than AC.

5. Prove that the ratio of the areas of similar triangles is equal to the ratio of the squares on corresponding sides.

A circle is described about a triangle ABC in which $AB = 6$ in., $BC = 9$ in., $AC = 4{\cdot}5$ in.; the tangent at A meets BC produced at D. Prove that $\triangle ABC : \triangle ABD = 7 : 16$.

6. From the top of a vertical cliff, 600 ft. high, the angle of depression of the top of a lighthouse is 10°. The lighthouse is known to be half a mile from the cliff. Find the height of the lighthouse and the angle of elevation of the top of the cliff from the foot of the lighthouse.

No. 72

1 By using the properties of parallel lines, prove that the sum of the angles of any triangle is two right angles

What other methods of proof are there ?

The sides AB, DC of a quadrilateral meet, when produced at X and the sides DA, CB meet, when produced, at Y, the angles AXD, DYC are bisected by lines meeting at O Prove that the angle XOY is equal to half the sum of one pair of opposite angles of the quadrilateral $ABCD$

2 Prove that the three medians of any triangle are concurrent Show that any two of the medians are greater than the third

3 Show that the triangle, having sides 15 cm, 13 cm and 7 cm, is obtuse angled

Calculate the length of the perpendicular let fall from the obtuse angle to the opposite side

4 A point P is taken $2\frac{1}{4}$ in from O the centre of a circle of $1\frac{1}{2}$ in radius Describe a circle of radius 2 in to pass through P and to touch the given circle Show that there are two such circles and measure the distance between their centres

5 If a point D be taken in the base AC of a triangle so that $AD : DC = AB : BC$, prove that DB bisects the angle ABC

If CE be drawn perpendicular to DB show that the angles ACE, BCE are respectively equal to half the difference and half the sum of the base angles BAC and BCA

6 Two circles with radii 3 5 in and 2 7 in are drawn with their centres 8 in apart Calculate the angle between their transverse common tangents

No. 73

1. From the extremities of a straight line AB are drawn parallel lines AX, BY, of the same length, but in opposite directions. Prove that AB bisects XY.

2. Construct a quadrilateral having each side and one diagonal equal to 1·5 in. Construct a triangle equal to it having one side of length 2·8 in.

3. Prove that any chord of a circle is bisected by the perpendicular let fall on it from the centre.

What is the locus of the middle points (i) of all chords passing through a fixed point, (ii) of all chords parallel to a fixed line.

4. Prove that the opposite angles of a cyclic quadrilateral are supplementary.

The arc BC of the circumcircle of a triangle ABC is bisected at P and PX, PY are drawn perpendicular to AB, AC respectively. Prove that $BX = CY$.

5. PQ is drawn parallel to the side BC of a triangle ABC. Prove that the circumcircles of the triangles APQ, ABC touch.

If PC and QB intersect at O, prove that the circumcircles of POQ and BOC touch.

6. AB is a diameter of a circle and OC is a radius perpendicular to AB; the tangents at A and B meet the tangent at C at P and Q respectively. The other tangent is drawn from X, the mid-point of AP, and produced to meet BQ at Y. Prove that QY is equal to the radius of the circle.

No. 74

1 Explain clearly what is meant by the converse of a proposition State the converses of the following propositions—

(i) If the angle ACB = the angle ADB, then the points A, B, C, D lie on a circle

(ii) If a point P is inside a triangle ABC, then the angle BPC is greater than the angle BAC

Discuss the truth of these converses

2 Draw a triangle ABC having $BC = 2\ 5$ in , $CA = 1\ 9$ in , $PB = 1\ 4$ in Take a point D in BC so that $BD = 2$ in Now through D a line that will bisect the triangle State and prove your construction

3 State, with proof, how to divide a line into two parts so that the rectangle contained by the whole line and one part is equal to the square on the other part

4 What is the locus of (i) the centres of all circles passing through two given points (ii) the centres of all circles touching two given intersecting straight lines

Being given two finite straight lines AB and CD, show how to draw two concentric circles having these lines as chords

5 Two circles with centres A and B cut at C, and the angle ACB is a right angle , the line of centres AB cuts the circles between A and B at D and E Prove that the angle DCE is half a right angle

6 If R and r denote the radii of the circumcircle and incircle of a triangle, prove, with or without trigonometry, $R = \dfrac{abc}{4\Delta}, r = \dfrac{\Delta}{s}, R^2 - 2Rr =$ sq of distance between the centres of the circles

[Δ denotes the area of the triangle s is the semi perimeter]

No. 75

1. Draw a line AB of length 1 in.; explain how to find, without using a ruler, a point C in AB produced such that BC is also 1 in. Prove your construction.

2. Give the enunciations of propositions that prove, (i) parallelogram, (ii) triangles, to be equal in area although not equal in all respects.

A point O is taken outside a parallelogram $ABCD$, so that it lies between AD and BC produced. Prove that $\triangle\ AOB = \triangle\ AOC + \triangle\ AOD$.

3. Prove that the diagonals of a parallelogram bisect one another.

Two parallelograms $ABDE$, $BCDE$ are on same base DE, and between the parallels CBA and DE; AD and CE intersect at F. Prove that BF, when produced, bisects DE.

4. Show how to inscribe a regular polygon of 15 sides in a given circle (i) using a protractor, (ii) not using a protractor.

5. In a trapezium $ABCD$, AB is parallel to DC and $AB = 3$ cm., $BC = 5$ cm., $CD = 9$ cm., $DA = 4$ cm., DA and CB are produced to meet in O. Calculate the lengths of OA and OB.

Verify by drawing a figure to scale.

6. Using the data of Question 5, calculate the size of the angle BCD and the area of the trapezium.

No. 76

1 The sides AB, CB of a triangle ABC are produced to D and E respectively so that $BD = AB$ and $BE = CB$ Prove that DE is equal and parallel to AC

2 Prove from the definition of a parallelogram that a parallelogram is bisected by either diagonal

Construct a parallelogram $ABCD$, having $BC = 2\ 3$ in, angle $ABC = 110°$, and area equal to that of a triangle DBC having angle $DBC = 49°$ and $DB = 2\ 8$ in

3 Prove, without using the proportion properties of parallels, that the straight line joining the mid points of two sides of a triangle is parallel to the third side

If P is the mid point of AB, Q of AC, and R of BC in a triangle ABC, prove that AR bisects PQ

4 Construct a triangle ABC having $BC = 3\ 2$ cm, angle $BAC = 100°$ and the median $AD = 1\ 5$ cm

5 If the opposite angles of a quadrilateral are supplementary, prove that the four vertices lie on a circle

The vertex A of a triangle ABC is joined to any point D in the side BC Prove that the centres of the circumcircles of the triangles ABC, ABD, ACD lie on a circle that passes through A

6 Explain how to find the fourth proportional to three given straight lines

By a geometrical construction find the value of $\dfrac{2\ 3 \times 1\ 8}{3\ 1}$

No. 77

1. The sides AB, CB of a triangle ABC are produced to D and E respectively, so that $BD = BC$ and $BE = BA$. Prove that AE is parallel to CD.

2. Prove that two sides of a triangle are greater than the third.

Take two points A and B on the same side of a straight line XY; show how to find a point C in XY so that $AC + CB$ is less than $AP + PB$ where P is any other point in XY.

3. Prove that the perpendicular to a chord from the centre of a circle bisects the chord.

From a point P, 7 cm. from O, the centre of a circle with radius 4·5 cm., draw a line so that the chord intercepted on it shall be 5·2 cm. long.

4. If from a point P outside a circle a tangent PT and a secant PAB be drawn, prove that $PT^2 = PA, PB$.

Find, without calculation, a point P in a line AB, length 3·5 in., produced so that rectangle $AP.PB$ may be of area 16 sq. in.

5. A point O is taken on the circumference of a circle and any three chords OA, OB, OC are drawn; outside the circle a triangle XYZ is drawn having YZ parallel to OA, ZX, parallel to OB, XY parallel to OC. Prove $AC : CB = XZ : ZY$.

6. If a polygon with n sides be inscribed in a circle of radius r, prove that the perimeter of the circle is $2nr \sin \dfrac{180^\circ}{n}$, and that its area is $\dfrac{n}{2} r^2 \sin \dfrac{360^\circ}{n}$. What are the corresponding values for n-sided polygon described about the circle ?

No 78

1 Define parallel straight lines and from your definition prove that, if a transversal cuts two parallel lines, the alternate angles are equal

Two circles cut at P and Q, through P a line APB is drawn cutting one circle at A and the other at B, through Q a line CQD is drawn cutting the circle APQ at C and the circle BPQ at D Prove that AC is parallel to BD

2 Prove that the diagonals of a rectangle are equal

A line PQ of constant length slides so that P is always on a fixed line OX, and Q on another fixed line OY which is perpendicular to OX Determine the locus of R the mid point of PQ

3 State the construction for making a triangle equal to a given pentagon $ABCDE$

Construct a regular pentagon with side 1 in, and make an isosceles triangle equal to it (use of protractor is allowed)

4 Prove that angles at the circumference of a circle subtended by the same chord are either equal or supplementary

Through the vertices of a triangle ABC any three lines YAZ, ZBX, XCY are drawn, forming a triangle XYZ Prove that the circumcircles of the triangle XBC, YCA, ZAB meet in a point

5 Describe a triangle ABC so that each side touches a given circle of radius 1 7 in, having the angle $A = 42°$, angle $B = 109°$

6 The vertex A of an equilateral triangle ABC is joined to any point R in BC, produced, and a point P is taken in RA, such that $RP \cdot RC = RB \cdot RA$ The line BP cuts AC at Q Prove that $AB^2 = AQ \cdot BR$

No. 79

1. Prove that the bisectors of adjacent angles are at right angles.

The centre of the inscribed circle of a triangle ABC is I and E is the centre of the R-scribed circle touching BC, not produced. Prove that B, I, C, E lie on a circle, the centre of which is on the circumcircle.

2. Construct a quadrilateral which shall be bisected by the diagonal AC and have $AB = 2$ in., $AC = 3$ in., angle $BAC = 32°$, and angle $BCD = 95°$.

3. Enunciate the geometrical theorem corresponding to the algebra formula : $a(a + b) = a^2 + ab$.

If a straight line AB is bisected at X and produced to Y, prove that $AX, AY = BX . BY + 2 BX^2$.

4. What are the respective loci of the centres of a circle satisfying the conditions : (i) To touch a given straight line and have a given radius ? (ii) To touch a given circle and have a given radius ?

Two equal circles move so that they touch each other and each touches one of two straight lines OX, OY, which are at right angles. Find the locus of R, their point of contact, (i) by drawing, (ii) by geometrical argument.

5. Prove that the perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.

Show that these perpendiculars bisect the angles formed by joining the feet of the perpendiculars.

6. Tangents CA, CB are drawn to a circle : any point P is taken on the circle and perpendiculars PX, PY, PZ drawn to CA, CB, AB respectively. Prove that $PX . PY = PZ^2$.

No. 80

1 State the rule for finding the sum of the angles of any rectilineal polygon

$ABCDEFG$ is a regular polygon with 7 sides, calculate the angles of the quadrilateral $CDFG$

2 A straight line AB is bisected at C, parallel lines AX, BY, CZ on the same side of AB meet another straight line at X, Y and Z Prove that $AX + BY = 2\ CZ$

What is the similar fact if AX, CZ are on the same side of AB and BY on the other side?

3 Prove Pythagoras's theorem, viz, that the square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the other two sides

A perpendicular ACD is drawn to a given line AB, C and D being any two points in the perpendicular, BA is produced and in the produced line points E and F are taken so that $AE = BC$ and $AF = BD$ Prove that $DE = CF$

4 Define a tangent to a circle as the limiting position of a secant Deduce from the angle property of a cyclic quadrilateral that an angle between a chord and a tangent at its extremity is equal to the angle in the segment on the other side of the chord

Two circles touch internally at O and a line cuts the larger circle at P and S the inner at Q and R Prove that the angle $POQ =$ the angle ROS

Is a similar statement true when the circles touch externally?

5 Two circles ABT, $ABCD$ cut at AB so that CD and AB, when produced, meet at P, and PT is a tangent to the circle ABT Prove that the circumcircle of the triangle CDT will touch the circle ABT

State the construction for drawing a circle to touch a given circle and to pass through two given points

6 Make a triangle ABC having sides $BC = 3\ 4$ cm, $BA = 2$ cm, $CA = 5$ cm Construct an angle CBD such that its sine shall be twice the sine of the angle BCA

Measure the angles and check your work by using tables

No. 81

1. Two triangles ABC, PQR have the sides BC and QR in the same straight line and are on the same side of BR; also $BC = QR$, $BA = QP$ and angles ABC and PQR are supplementary. Prove that AP is parallel to BC and QR.

2. If a parallelogram is held under water, prove that in every position the sum of the depths of the four corners is equal to four times the depth of the intersection of the diagonal.

3. Any point P is taken in the diagonal AC of a parallelogram $ABCD$; through P lines HPK, LPM are drawn parallel to the sides of the parallelogram. meeting AB, BC, CD, DA in M, K, L, H respectively. Prove that the parallelograms $PMBK$, $PLDM$ are equal in area.

Hence construct a rectangle with one side 2·3 in. long, and of area 5 sq. in. Verify by measurement.

4. Explain how to draw common tangents to two given circles. State when there are 4, 3, 2, 1, 0 common tangents respectively.

Two circle centres A and B have radii 3 cm. and 2 cm. respectively, and $AB = 7$ cm. Draw a line $PQRS$ cutting circle A at P and Q, circle B at R and S, so that $PQ = 5$ cm., $RS = 3$ cm.

5. Any point P is taken on the circumcircle of a triangle ABC, and perpendiculars PH, PK, PL are drawn to the sides BC, CA, AB, produced when necessary. Prove that H, K, L are in the same straight line.

6. The dimensions of a rectangular room are length 20 ft., width 15 ft., height 12 ft. Find, by drawing and calculation, the length of a diagonal and the angle it makes with a diagonal of the floor.

No. 82

1 ABC is any triangle, show that any number of equilateral triangles can be drawn having their three sides passing respectively through A, B and C

2 Prove that the opposite sides and angles of a parallelogram are equal

Prove that the figure formed by the intersection of the bisector of the angles of a parallelogram is a rectangle

3 What is the complete locus of points equidistant from two given intersecting straight lines? Prove your answer

Draw two lines POQ, ROS intersecting at an angle of 50° Find the locus of a point which moves so that its distance from POQ is always 2 cm greater than its distance from ROS

4 Divide a straight line into two parts so that the square on one part may be equal to the rectangle contained by the whole line and the other part

Divide a straight line AB, 3 in long, at a point O so that $AB^2 + BC^2 = 3\ AC^2$

5 If K is the orthocentre of a triangle ABC, prove that $AK = BC \cot A$

A is a fixed point on the circumference of a circle, and PQ is a variable chord of constant length What is the locus of the orthocentre of the triangle APQ?

6 The side BA of a triangle ABC is produced and the exterior angle so formed is bisected by a line which cuts the circumcircle at D and BC produced at E Prove that $AB\ AC = AD\ AE$

No. 83

1. Draw two triangles ABC, PQR, having two sides AB, AC equal respectively to PQ, PR, having also one angle of ABC equal to one angle of PQR, and yet the triangles are not congruent.

Two lines meet at A. With centre A an arc is drawn cutting the one line at P, the other at Q, and with centre A, another arc is drawn cutting the former line at X and the latter at Y. If PY, QX intersect at Z, prove that AZ bisects the angle at A.

2. If a point O is taken inside a triangle ABC, prove that the angle BOC is greater than the angle BAC.

Show that the converse is not necessarily true.

3. Illustrate by geometrical figures the algebraic formula—

(i) $a^2 - b^2 = (a + b)(a - b)$;

(ii) $(a + b)^2 - (a - b)^2 = 4ab$.

4. Prove that every parallelogram inscribed in a circle must be a rectangle.

If the middle points of the sides of a quadrilateral lie on a circle, prove that the diagonals are at right angles.

5. If two triangles have an angle of the one equal to an angle of the other, and the sides about those angles proportional, prove that the triangles are similar.

In one of the sides AB of a triangle ABC, having $AB = AC$, a point D is taken such that $CB = CD$. Prove that $AB . BD = BC^2$.

6. When the sun is at altitude 50° at noon, find (i) the length of the shadow of a post 46 ft. high, (ii) the area of the shadow of a wall, the length of which is 90 ft., the height 7 ft., and the direction north-east.

No. 84

1 If two sides of a triangle are equal, prove that the opposite angles are also equal

If the point P in the side BC of a triangle which is equidistant from A and B is also the mid point of BC, prove that the triangle is right-angled

2 In what kinds of parallelogram (i) are the diagonals at right angles? (ii) do the diagonals bisect the angles?

In a trapezium $ABCD$ the non parallel sides AD and BC are equal Prove that A, B, C, D lie on a circle

3 Divide a straight line AB, 2 7 in long, at a point P, so that $AB^2 = BP^2 + 2\, AB.\, BP$

4 Prove that the line joining the centre of a circle to the middle point of a chord is perpendicular to the chord

Two circles cut at X, PXQ is a line through X, meeting one circle at P and the other at Q Prove that PQ is greatest when it is parallel to the line of centres

5 If I is the centre of the inscribed circle of a triangle, prove that the angle $BIC = 90 + \frac{A}{2}$

A Q P

Describe a triangle ABC having A as one vertex, P the point where the bisector of the angle B meets AC, and Q the point where the bisector of the angle C meets AB

6 Tangents PA, PB are drawn to a circle with centre O, and any point X is taken on AB produced The line at right angles to OX at X meets AP, PB, produced at Y and Z respectively Prove that $XY = XZ$

No. 85

1. Construct a trapezium $ABCD$ having AB parallel to DC and $AB = 3{\cdot}7$ cm., $BC = 2{\cdot}5$ cm., $CD = 5{\cdot}6$ cm., angle $CDA = 70°$. Take the necessary measurement and find its area.

2. Prove that two triangles are equal in area if they have two sides of the one equal to two sides of the other, each to each, and the included angles together equal to two right angles.

3. Give, without proof, a geometrical illustration of the algebraical identity $(a-b)^2 = a^2 + b^2 - 2ab$.

If one angle of a triangle is one-third of two right angles, prove that the square on the opposite side is less than the sum of the squares on the sides containing that angle, by the rectangle contained by these two sides.

4. Show that two tangents can be drawn to a circle from an external point and that the parts of them intercepted between the point and the circle are equal.

If the inscribed circle of the triangle ABC touches AB at R, prove that $2\,AR = AB + AC - BC$.

5. Prove that equal chords of a circle subtend equal angles at the centre, and that the angles they subtend at a point on the circumference are either equal or supplementary.

Two equal circles ABC and ADC cut at A and B, and $AB = AC$. If BC meets the other circle at D, prove that D is inside the circle ABC.

6. If I is the centre of the inscribed circle of a triangle ABC, prove that $AI : ID = BA + AC : BC$ where D is the point of intersection of AI and BC.

No. 86

1 State and prove a construction for drawing a line perpendicular to a given line from a given point outside it

A and B are given points on opposite sides of a given line XY Find a point C in XY such that the angle ACX is equal to the angle BCX.

2 Prove that the quadrilateral formed by joining the middle points of the sides of a rectangle is a rhombus, and that its area is half that of the rectangle

3 If P and Q are points on the same side of AB such that the angles APB, AQB are equal, prove that the four points A, P, Q, B lie on a circle

Equilateral triangles APB, AQC are described on the sides AB, AC of a triangle ABC, falling outside that triangle If BQ, CP intersect at R, prove that A, C, Q R lie on a circle

4 In any triangle prove that the difference of the squares on two sides is equal to the rectangle contained by the third side, and the projection on it of the median bisecting it

If A, B, C, D are four points such that $AC^2 - BC^2 = AD^2 - BD^2$, prove that CD is perpendicular to AB

5 Explain how to inscribe a circle in a given triangle Show that the radius is equal to the area of the triangle divided by the semi perimeter

6 Tangents PA, PB are drawn from a point P to a circle with centre O, and PO cuts AB at C Prove that OC CP $= AC^2$ If XY is any other chord through C, prove that OP bisects the angle XPY

No. 87

1. If two triangles have their sides equal, each to each, prove that they are equal in all respects.

On the side AB of a square $ABCD$ an equilateral triangle ABE is described, and F is the mid-point of CD. Prove that EF is at right angles to CD.

2. Construct a quadrilateral $ABCD$ having $AB = 1{\cdot}7$ in., diagonal $AC = 2{\cdot}3$ in., angle $BAC = 80°$, angle $BCD = 90°$, angle $ADC = 90°$. Make an isosceles triangle equal to the quadrilateral, having AC as base.

3. A cone stands on a base of radius 5 in. and its slant side is of length 13 in.; find, by drawing or calculation, the radius of the largest sphere that can be covered by the cone.

4. In a triangle ABC the angle at B is obtuse, AD is let fall perpendicular to BC, meeting it at D. Prove that D must be outside the triangle, and that B is between C and D. Prove $AB^2 = AC^2 + BC^2 - 2\,BC\,.\,CD$.

5. If a straight line PQ cuts the sides AB, AC of a triangle in the same ratio, prove that PQ is parallel to BC.

Take any point P in the side AB of a triangle ABC, and show how to draw a line through P to meet BC produced at Q, such PQ may be bisected by AC.

6. A regular pyramid has a square base; the faces are isosceles triangles with base angles each equal to 70°. Find the inclination of the faces to the base (i) by drawing, (ii) by trigonometrical calculation.

No 88

1 Make an angle XOY equal to $55°$, on OX measure $OA = 3$ cm, $OB = 7$ cm From A and B draw two equal straight lines AC and BC to meet on OY Measure them

2 If the intercepts made by three parallel lines on any one transversal are equal prove that the intercepts made on any other transversal are also equal

Perpendiculars BP, CQ are let fall from the vertices B and C of a triangle ABC upon any line through A between AB and AC Prove that D, the mid point of BC, is equidistant from P and Q

3 Explain how to describe an equilateral triangle about a given circle

ABC is an equilateral triangle described about a circle with centre O, and D is the point of contact of BC with the circle If AD cuts the circle at E prove that $AE = EO = OD$

4 Divide a straight line AB of length 7 cm into two parts at C so that the rectangle AC CB may be of area 9 sq cm

Solve the equation $x^2 - 7x + 9 = 0$ by a geometrical construction

5 ABC is an equilateral triangle inscribed in a circle, D is a point on the circumference in the minor arc BC On BD an equilateral triangle BED is described, so that E falls inside the circle Prove that A, E, D are in same straight line

6 With the data in Question 1, find the lengths of AC and BC with the aid of trigonometry

No. 89

1. Two triangles ABC, DBC on the same side of the base BC have $AB = DC$ and $AC = DB$; AC and DB intersect in E. Prove that $EB = EC$.

2. Prove that, if a transversal cuts two parallel lines, the two interior angles on the same side are supplementary.

The bisectors of the angles B and C of a parallelogram $ABCD$ meets at a point P, and the length of AB is 6 in. If the point P lies in the side AD, find the length of BC.

3. Construct a triangle ABC having $AB = 3$ in., and the angles A, B, C in the ratio $2:3:4$. Make a square equal to it without any further calculation. Measure the side of the square to the nearest fortieth of an inch.

4. At a point P on a circle of radius 2 in., a tangent PQ is drawn of length 3 in. Describe a circle passing through Q, touching the first circle and having its centre on PQ.

5. The side AC of a triangle ABC is produced both ways to D and E, so that $AD = AB$ and $CE = CB$, and a circle with centre O is described to pass through B, D, and E. Prove that OB bisects the angle ABC.

6. Prove Ptolemy's theorem, viz. the sum of the rectangles contained by the opposite sides of a cyclic quadrilateral is equal to the rectangle contained by the diagonals.

An equilateral triangle ABC is inscribed in a circle; P is a point on the minor arc BC. Prove that $PA = PB + PC$.

No. 90

1. Prove that, in any triangle, the side opposite the greater of two of the angles is greater than the side opposite the less.

Points P and Q are taken on the sides AB, AC respectively of a triangle ABC. If the angle A is the greatest angle, prove that PQ must be less than BC.

2. Describe a triangle ABC, being given that $BC = 4{\cdot}7$ cm., angle $BCA = 79°$, and that the radius of the circumcircle is 3 cm. Prove your construction to be correct.

3. On the same side of AB an equilateral triangle APB and a square $ABCD$ are described Prove that P falls inside the square.

If BP produced meets CD at Q, prove that the angle DPQ is half a right angle.

4. Two equal circles have centres A and B; a line parallel to AB cuts the A circle at P and Q, and the B circle at H and K, so that PQ and HK are in the same sense. Prove that $PH = QK = AB$.

Show how to draw a circle to touch two given parallel straight lines and to pass through a given point between them.

5. Prove that the bisector of the vertical angle of a triangle divides the opposite side into segments which have the same ratio as the sides containing the bisected angle.

AD is a median of the triangle ABC, the angles ADB, ADC are bisected by lines meeting AB, AC, at E and F respectively. Prove that EF is parallel to BC.

6. PQ and XY are parallel lines 4 cm. apart, and A is a point between them 1·5 cm. from XY. Construct a square $ABCD$ having B on XY and D on PQ; prove the accuracy of the construction. Measure the angle ABX and verify by trigonometry.

No. 91

1. In a triangle ABC the sides AB and AC are equal; the angles ABC, ACB are bisected by straight lines meeting at D. Prove that AD produced bisects BC.

2. Prove that two sides of any triangle are greater than the third side.

Prove that any two medians of a triangle are greater than the third, and that the sum of the medians is less than the perimeter of the triangle.

3. Take a point O 3 cm., from a line XY. Draw the locus of the middle points of lines drawn from O to XY. Prove the truth of your construction.

4. Prove, by a geometrical figure, that the difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.

A point O is taken in the base BC of an isosceles triangle ABC; prove that $AB^2 - AO^2 = BO \cdot OC$.

5. Two circles intersect at A; a line PAQ meets one of the circles at P, the other at Q, and bisects an angle between the tangents at A. Prove that $PA : AQ$ equals the ratio of the radii.

6. Prove that the area of any triangle is equal to the rectangle contained by two sides multiplied by the sine of the included angle.

A, B, C are three fixed points in order on a straight line and O is any point outside the line. Prove that:

(i) $OB \sin AOB : OC \sin AOC$ is the same for all positions of O

(ii) $\dfrac{\sin AOB}{OC} + \dfrac{\sin BOC}{OA} = \dfrac{\sin COA}{OB}$.

No 92

1 If two parallelograms have two adjacent sides of the one equal to two adjacent sides of the other, each to each and one angle of the one equal to one angle of the other prove that the parallelograms will coincide if one is superposed on the other

2 Draw a triangle ABC having $BC = 3\ 2$ cm $CA = 2\ 1$ cm $AB = 3$ cm On a base PQ of length 2 in construct without calculation a triangle PQR equal in area to the triangle ABC

3 A triangle PBC is drawn on a given line BC 2 1 in long Find the locus of P if $PB^2 - PC^2 = 7$ of a sq in

4 Prove that if two circles touch, either internally or externally the point of contact and the two centres are in the same straight line

Circles are drawn to touch a given circle and a given straight line Show that part of the complete locus of their centres is the same as the locus of a point equidistant from a certain fixed point and from a certain fixed line

5 Two circles intersect at A show how to draw a straight line through A so that the circles shall intercept equal chords on that line

Show how to draw a square so that opposite sides may pass through two given points A and B and the diagonals may intersect at another given point C

6 The tangent at A to the circumcircle of a triangle ABC meets BC produced at P Prove that—

$$PB\ \ PC = AB^2\ \ AC^2$$

No. 93

1. ABC is a triangle, right-angled at B; $AB = 1{\cdot}7$ in., and angle A is 57°. Draw the triangle and measure AC, BC.

Confirm your measurements by calculating these lengths by using trigonometrical tables.

2. Prove that the opposite sides and angles of a parallelogram are equal.

The vertex A of a parallelogram $ABCD$ is joined to any point E in the side CD; any point F is taken in EA, and EA is produced to G so that $AG = EF$. Prove that the parallelogram $FGHB$ is equal in area to the parallelogram $ABCD$.

3. If there are two straight lines, one of which is divided into two parts, show by a figure that the rectangle contained by the undivided line and a part of the divided line is equal to the rectangle contained by the two lines diminished by the rectangle contained by the undivided line and the other part of the divided line.

Hence give a geometrical demonstration of the algebraical identity $a(b-c) + b(c-a) = c(b-a)$.

4. In a triangle ABC, the perpendiculars from B and C on the opposite sides intersect at D, and meet the circumcircle in E and F. Prove that A is the centre of the circumcircle of the triangle DEF.

5. Prove that the locus of a point whose distances from two given points are in a constant ratio is a circle.

6. From a point P in the side BC of a triangle ABC, PQ is drawn parallel to AB to meet AC in Q, and PR parallel to AC to meet AB in R, and QR is produced to meet BC at S. Prove that SP is the mean proportional between SB and SC.

No. 94

1 Two lines OA and OB are respectively at right angles to OX and OY, determine whether the bisector of an angle between OA and OB also bisects one of the angles between OX and OY

2 The diagonals of a parallelogram $ABCD$ intersect at O, and a line POQ cuts AB at P and DC at Q, prove that PQ is bisected at O

Draw a triangle ABC having $BC = 2\cdot3$ cm, $CA = 1\cdot5$ cm, $AB = 1\cdot7$ cm Construct a rhombus so that two adjacent sides contain an angle 60° and pass through A and B respectively, and the diagonals intersect at C Prove your construction

3 Show how to produce a given straight line so that the rectangle contained by the whole line thus produced, and the part produced, is equal to the square on the original line

4 Prove that an angle at the centre of a circle is double any angle at the circumference standing on the same arc

Two circles, PQX and PQY, cut orthogonally (i e their tangents at a point of intersection are at right angles) at P and Q, PX and PY are two chords at right angles Prove that X, Q, Y are in the same straight line

5 If two triangles are equiangular, prove that their corresponding sides are in the same ratio

In Question 4, prove that the ratio of PX to QY is equal to the ratio of the diameters of the circles

6 An observer, whose eye C is 5 6 ft from the ground and 90 ft from a vertical tower AB, finds that the tower subtends at his eye an angle of 59° Calculate the height of the tower

No. 95

1. $ABCD$ is a square, with A as centre a circle is described cutting AB at E and AD at F. Prove that the line through A at right angles to DE bisects BF.

2. ACB is an isosceles right-angled triangle with AC equal to BC; at any point P in AB, nearer to B than A, a perpendicular is erected meeting BC at Q. Prove that $PQ = PB$.

Hence show how to divide a given line AB so that the sum of the squares on the two parts may be as small as possible.

3. Deduce from the construction of Question 2 that, if a straight line be divided into two equal parts and also into two unequal parts, the sum of the squares on the two unequal parts is equal to twice the sum of the squares on half the line and the line between the points of section.

4. Prove that angles in the same segment of a circle are equal to one another.

If the segment is smaller than a semi-circle, prove this proposition without using reflex angles.

5. Give a geometrical proof of the proposition that if four straight lines are in proportion, the rectangle contained by the means is equal to the rectangle contained by the extremes.

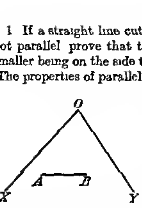
Prove that either of the equal sides of an isosceles triangle is a mean proportional between the diameter of the circumscribing circle, and the perpendicular from the vertex.

6. Describe a regular pentagon with a side of 1 in., the use of protractor being allowed. State how it might be done without using a protractor. Calculate the distance of a vertex from the opposite side.

Hence, or otherwise, show that if a pentagon and a hexagon are described on the same side of the same base, the pentagon will fall entirely inside the hexagon.

No. 96

1 If a straight line cuts two other straight lines, which are not parallel prove that the alternate angles are unequal, the smaller being on the side towards which the lines tend to meet (The properties of parallel lines are not to be assumed)



2 Explain, with proof, how to draw a parallelogram $ABCD$, having C on OY and D on OX

3 If from a point O within a triangle ABC, perpendiculars OX, OY, OZ are let fall on BC, CA, AB respectively, prove that $AZ^2 + BX^2 + CY^2 = AY^2 + CX^2 + BZ^2$

Where must O be taken so that $AZ^2 + ZB^2 + BX^2 + XC^2 + CY^2 + YA^2$ may be as small as possible ?

4 Construct a triangle ABC being given $BC = 2$ in , angle $BAC = 130°$, and the median $AD = 7$ in

5 If a point O is taken inside a triangle ABC, and AO, BO, CO are produced to meet the opposite sides at P, Q R respectively, prove that $\frac{BP}{PC} \frac{CQ}{QA} \frac{AR}{RB} = 1$ [This is Ceva's Theorem]

6 A circle passing through A, a vertex of the rectangle $ABCD$, cuts AB at P, AC at Q AD at R prove that—

(i) $AB \ AP + AD \ AR = AC \ AQ$

(ii) $AB \ BP + AD \ DR = AC \ CQ$

No. 97

1. Prove that the triangle formed by joining the mid-points of the sides of an isosceles triangle is also isosceles.

2. Construct a rectangle having two of the sides each 1·4 in. long, and a diagonal 3·1 in. long. Measure the other sides to the nearest tenth of an inch.

3. A and B are two points on the same side of a line XY; show how to find a point L in XY such that $LA + LB$ is a minimum.

Prove that the triangle of minimum perimeter that can be inscribed in a given triangle is formed by joining the feet of the perpendiculars let fall from the vertices on the opposite sides.

4. The bisector of the angle A of a triangle ABC cuts the perpendicular bisector of the side BC at D. Prove that A, B, C, D are concyclic.

If P is any point not on the circumcircle of the triangle ABC, prove that the line joining the centres of the circumcircles of the triangles ADP and BCD bisects AD.

5. Show that four circles can, in general, be found to touch each of three given straight lines; and prove that the centre of any one of these circles is the orthocentre of the triangle formed by the centres of the other three circles.

6. Prove that if two triangles are equiangular, their areas are proportional to the squares on corresponding sides.

Construct (i) a square of area 15 sq. in., (ii) a square equal in area to an equilateral triangle of side 1 in. Hence, construct an equilateral triangle of area 15 sq. in. Verify by calculation.

No 98

1 A brass weight consists of a cylindrical portion surmounted by a cone, the top of the cylinder being the base of the cone The radius of the common section is 1 1 in, the total height is 5 in, the slant height of the cone is 3 in Draw the elevation (i e side view) of the brass weight

2 Show how to draw a line from a point P in the side BC of a triangle ABC, which will bisect the triangle

3 If the diagonal AC of a rhombus $ABCD$ is produced to any point P, prove that $PA \cdot PC = PB^2 - AB^2$

4 Draw a circle of radius 3 cm, and in it inscribe a triangle having angles 33° and 66°

About the same circle describe a triangle, equiangular to the former, the sides of which touch the circle

In each triangle measure the sides opposite the angles 66° Verify by trigonometrical calculation

5 Prove that the rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the rectangles contained by pairs of opposite sides

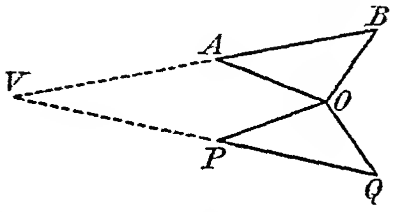
A point D is taken in the side BC produced, of an equilateral triangle ABC, and on AD, on the opposite side to C, an equilateral triangle ADE is drawn Prove that $CE = BD$

6 Prove the formula $c^2 = a^2 + b^2 - 2\,ab \cos C$, considering the three cases (i) C acute, (ii) C obtuse, (iii) C a right angle

Find all the angles of the triangle whose sides are 6 8 cm, 4 2 cm, 4 0 cm

No. 99

1. Explain the method of proof by *reductio ad absurdum.* Use this method to prove that if two angles of a triangle are equal, the opposite sides are also equal.



OAB, OPQ are two triangles having $OA = OP$, $OB = OQ$ and $AB = PQ$; BA and QP are produced to meet at V as in the figure. Prove that $VB = VQ$.

2. Without using any proposition about areas other than the equality of congruent triangles, prove that parallelograms on the same base and between the same parallels are equal in area.

Make a parallelogram $ABCD$ having $AB = 2$ in., $BC = 3$ in., and the diagonal $BD = 4$ in. Construct a parallelogram equal in area to $ABCD$, having two sides each equal to 2·5 in., and two angles each equal to 110°.

3. State a necessary condition involving angles about four points that lie on a circle.

In a triangle ABC, AP, BQ are perpendiculars from A and B on the opposite sides, E and F are the mid-points of AC and AB respectively. Prove that P, Q, E, F lie on a circle.

4. A and B, the centres of two circles of radii 2·3 cm. and 3 cm. respectively, are 6 cm. apart. Find all the points P such that the tangents from P to the A circle are 3 cm. long, and those to the B circle are 4 cm. long.

5. With centre C on the circumference of a given circle another circle is described cutting the former at A and B; through A a line is drawn cutting the first circle at P, and the second at Q. Prove that $PB = PQ$.

6. Four circles can be described each touching the sides or sides produced of a triangle. Prove that the circumcircles of the four triangles formed by the centres of these circles are equal.

No. 100

1 Prove that an exterior angle of a triangle is equal to the sum of the interior opposite angles

Find a point X in the side BC of a triangle ABC, such that XD, the perpendicular on AB, is equal to XC State and prove your construction

2 The diagonal AC of a quadrilateral $ABCD$ is produced through C to any point E If $AB = AD$ and $CB = CD$, prove that EA bisects the angle BED

3 Prove that in equal circles (or the same circle) equal angles at the circumferences stand on equal chords

A point P moves so that the angle APB is constant in size, AB being fixed in position and magnitude In any position of P, AK is let fall perpendicular to BP and BL perpendicular to AP Prove that KL is of constant length

4 Find a point P on the circumcircle of a triangle ABC, so that, if AP is produced to meet BC produced at Q, AQ may be bisected at P Prove the truth of your construction

5 If at a point A on a circle a chord AB is drawn, and another line AX is drawn on the side of the minor segment, such that the angle XAB is equal to the angle in the major segment, prove that AX is a tangent to the circle

AB, BC, CD are three equal straight lines such that BA and CD are parallel on opposite sides of BC The circumcircle of the triangle ABC cuts CD at E Prove that the rectangle AB, DE is equal to the square on AC

6 If two triangles have an angle of the one equal to an angle of the other, and the sides about those angles proportional, prove that the triangles are similar

ACB, ADB are two right angled triangles on the same side of the hypotenuse AB, and $AC = BD = \frac{1}{2}AB$ O is the mid point of AB, and AC is produced to E, making AE equal to AO If BE cuts OD at F, prove that $OF = BF$

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